mathematics

Article

Truncated Fubini Polynomials

Ugur Duran [1,*] and Mehmet Acikgoz [2]

1 Department of the Basic Concepts of Engineering, Faculty of Engineering and Natural Sciences, Iskenderun Technical University, TR-31200 Hatay, Turkey
2 Department of Mathematics, Faculty of Science and Arts, University of Gaziantep, TR-27310 Gaziantep, Turkey; acikgoz@gantep.edu.tr
* Correspondence: mtdrnugur@gmail.com

Received: 22 April 2019; Accepted: 9 May 2019; Published: 15 May 2019

Abstract: In this paper, we introduce the two-variable truncated Fubini polynomials and numbers and then investigate many relations and formulas for these polynomials and numbers, including summation formulas, recurrence relations, and the derivative property. We also give some formulas related to the truncated Stirling numbers of the second kind and Apostol-type Stirling numbers of the second kind. Moreover, we derive multifarious correlations associated with the truncated Euler polynomials and truncated Bernoulli polynomials.

Keywords: Fubini polynomials; Euler polynomials; Bernoulli polynomials; truncated exponential polynomials; Stirling numbers of the second kind

MSC: Primary 11B68; Secondary 11B83, 11B37, 05A19

1. Introduction

The classical Bernoulli and Euler polynomials are defined by means of the following generating functions:

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}=\frac{t}{e^t-1}e^{xt}\qquad(|t|<2\pi) \tag{1}$$

and:

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt}\qquad(|t|<\pi), \tag{2}$$

see [1–10] for details about the aforesaid polynomials. The Bernoulli numbers B_n and Euler numbers E_n are obtained by the special cases of the corresponding polynomials at $x=0$, namely:

$$B_n(0):=B_n \text{ and } E_n(0):=E_n. \tag{3}$$

The truncated exponential polynomials have played a role of crucial importance to evaluate integrals including products of special functions; cf. [11], and also see the references cited therein. Recently, several mathematicians have studied truncated-type special polynomials such as truncated Bernoulli polynomials and truncated Euler polynomials; cf. [1,4,7,9,11,12].

For non-negative integer m, the truncated Bernoulli and truncated Euler polynomials are introduced as follows:

$$\sum_{n=0}^{\infty} B_{m,n}(x) \frac{t^n}{n!} = \frac{\frac{t^m}{m!}}{e^t - \sum_{j=0}^{m-1} \frac{t^j}{j!}} e^{xt} \quad \text{(cf. [1])} \tag{4}$$

and:

$$\sum_{n=0}^{\infty} E_{m,n}(x) \frac{t^n}{n!} = \frac{2\frac{t^m}{m!}}{e^t + 1 - \sum_{j=0}^{m-1} \frac{t^j}{j!}} e^{xt} \quad \text{(cf. [7])}. \tag{5}$$

Upon setting $x = 0$ in (4) and (5), the mentioned polynomials ($B_{m,n}(x)$ and $E_{m,n}(x)$), reduce to the corresponding numbers:

$$B_{m,n}(0) := B_{m,n} \text{ and } E_{m,n}(0) := E_{m,n} \tag{6}$$

termed as the truncated Bernoulli numbers and truncated Euler numbers, respectively.

Remark 1. *Setting $m = 0$ in (4) and $m = 1$ (5), then the truncated Bernoulli and truncated Euler polynomials reduce to the classical Bernoulli and Euler polynomials in (1) and (2).*

The Stirling numbers of the second kind are given by the following exponential generating function:

$$\sum_{n=0}^{\infty} S_2(n,k) \frac{t^n}{n!} = \frac{\left(e^t - 1\right)^k}{k!} \quad \text{(cf. [2–5,7,8,10,13])} \tag{7}$$

or by the recurrence relation for a fixed non-negative integer ζ,

$$x^{\zeta} = \sum_{\mu=0}^{\zeta} S_2(\zeta, \mu) (x)_{\mu}, \tag{8}$$

where the notation $(x)_{\mu}$ called the falling factorial equals $x(x-1)\cdots(x-\mu+1)$; cf. [2–5,7–10,13], and see also the references cited therein.

The Apostol-type Stirling numbers of the second kind is defined by (cf. [8]):

$$\sum_{n=0}^{\infty} S_2(n,k:\lambda) \frac{t^n}{n!} = \frac{\left(\lambda e^t - 1\right)^k}{k!} \quad \left(\lambda \in \mathbb{C}/\left\{1\right\}\right). \tag{9}$$

The following sections are planned as follows: the second section includes the definition of the two-variable truncated Fubini polynomials and provides several formulas and relations including Stirling numbers of the second kind with several extensions. The third part covers the correlations for the two-variable truncated Fubini polynomials associated with the truncated Euler polynomials and the truncated Bernoulli polynomials. The last part of this paper analyzes the results acquired in this paper.

2. Two-Variable Truncated Fubini Polynomials

In this part, we define the two-variable truncated Fubini polynomials and numbers. We investigate several relations and identities for these polynomials and numbers.

We firstly remember the classical two-variable Fubini polynomials by the following generating function (cf. [2,3,5,6,10,13]):

$$\sum_{n=0}^{\infty} F_n(x,y) \frac{t^n}{n!} = \frac{e^{xt}}{1 - y\left(e^t - 1\right)}. \tag{10}$$

When $x = 0$ in (10), the two-variable Fubini polynomials $F_n(x,y)$ reduce to the usual Fubini polynomials given by (cf. [2,3,5,6,10,13]):

$$\sum_{n=0}^{\infty} F_n(y) \frac{t^n}{n!} = \frac{1}{1 - y(e^t - 1)}. \tag{11}$$

It is easy to see that for a non-negative integer n (cf. [2]):

$$F_n\left(x, -\frac{1}{2}\right) = E_n(x), \quad F_n\left(-\frac{1}{2}\right) = E_n \tag{12}$$

and (cf. [3,5,6,10,13]):

$$F_n(y) = \sum_{\mu=0}^{n} S_2(n,\mu)\, \mu! y^{\mu}. \tag{13}$$

Substituting y by 1 in (11), we have the familiar Fubini numbers $F_n(1) := F_n$ as follows (cf. [2,3,5,6,10,13]):

$$\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} = \frac{1}{2 - e^t}. \tag{14}$$

For more information about the applications of the usual Fubini polynomials and numbers, cf. [2,3,5,6,10,13], and see also the references cited therein.

We now define the two-variable truncated Fubini polynomials as follows.

Definition 1. *For non-negative integer m, the two-variable truncated Fubini polynomials are defined via the following exponential generating function:*

$$\sum_{n=0}^{\infty} F_{m,n}(x,y) \frac{t^n}{n!} = \frac{\frac{t^m}{m!} e^{xt}}{1 - y\left(e^t - 1 - \sum_{j=0}^{m-1} \frac{t^j}{j!}\right)}. \tag{15}$$

In the case $x = 0$ in (15), we then get a new type of Fubini polynomial, which we call the truncated Fubini polynomials given by:

$$\sum_{n=0}^{\infty} F_{m,n}(y) \frac{t^n}{n!} = \frac{\frac{t^m}{m!}}{1 - y\left(e^t - 1 - \sum_{j=0}^{m-1} \frac{t^j}{j!}\right)}. \tag{16}$$

Upon setting $x = 0$ and $y = 1$ in (15), we then attain the truncated Fubini numbers $F_{m,n}$ defined by the following Taylor series expansion about $t = 0$:

$$\sum_{n=0}^{\infty} F_{m,n} \frac{t^n}{n!} = \frac{\frac{t^m}{m!}}{2 + \sum_{j=m}^{\infty} \frac{t^j}{j!}}. \tag{17}$$

The two-variable truncated Fubini polynomials $F_{m,n}(x,y)$ cover generalizations of some known polynomials and numbers that we discuss below.

Remark 2. *Setting $m = 0$ in (15), the polynomials $F_{m,n}(x,y)$ reduce to the two-variable Fubini polynomials $F_n(x,y)$ in (10).*

Remark 3. *When $m = 0$ and $x = 0$ in (15), the polynomials $F_{m,n}(x,y)$ become the usual Fubini polynomials $F_n(y)$ in (11).*

Remark 4. *In the special cases $m = 0$, $y = 1$, and $x = 0$ in (15), the polynomials $F_{m,n}(x,y)$ reduce to the familiar Fubini numbers F_n in (14).*

We now are ready to examine the relations and properties for the two-variable Fubini polynomials $F_n(x,y)$, and so, we firstly give the following theorem.

Theorem 1. *The following summation formula:*

$$F_{m,n}(x,y) = \sum_{k=0}^{n} \binom{n}{k} F_{m,k}(y)\, x^{n-k} \tag{18}$$

holds true for non-negative integers m and n.

Proof. By (15), using the Cauchy product in series, we observe that:

$$\begin{aligned} \sum_{n=0}^{\infty} F_{m,n}(x,y) \frac{t^n}{n!} &= \frac{\frac{t^m}{m!}}{1 - y\left(e^t - 1 - \sum_{j=0}^{m-1} \frac{t^j}{j!}\right)} e^{xt} \\ &= \sum_{n=0}^{\infty} F_{m,n}(y) \frac{t^n}{n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} F_{m,k}(y)\, x^{n-k} \frac{t^n}{n!}, \end{aligned}$$

which provides the asserted result (18). □

We now provide another summation formula for the polynomials $F_{m,n}(x,y)$ as follows.

Theorem 2. *The following summation formulas:*

$$F_{m,n}(x+z,y) = \sum_{k=0}^{n} \binom{n}{k} F_{m,k}(x,y)\, z^{n-k} \tag{19}$$

and:

$$F_{m,n}(x+z,y) = \sum_{k=0}^{n} \binom{n}{k} F_{m,k}(y)\, (x+z)^{n-k} \tag{20}$$

are valid for non-negative integers m and n.

Proof. From (15), we obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{m,n}(x,y)\frac{t^n}{n!} &= \frac{\frac{t^m}{m!}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}e^{(x+z)t} \\
&= \frac{\frac{t^m}{m!}e^{xt}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}e^{zt} \\
&= \sum_{n=0}^{\infty} F_{m,n}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty} z^n\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}F_{m,k}(x,y)\,z^{n-k}\frac{t^n}{n!}
\end{aligned}$$

and similarly:

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{m,n}(x,y)\frac{t^n}{n!} &= \frac{\frac{t^m}{m!}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}e^{(x+z)t} \\
&= \sum_{n=0}^{\infty} F_{m,n}(y)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x+z)^n\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}F_{m,k}(y)\,(x+z)^{n-k}\frac{t^n}{n!}
\end{aligned}$$

which yield the desired results (19) and (20). □

We here define the truncated Stirling numbers of the second kind as follows:

$$\sum_{n=0}^{\infty} S_{2,m}(n,k)\frac{t^n}{n!} = \frac{\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)^k}{k!}. \tag{21}$$

Remark 5. *Upon setting $m=0$ in (21), the truncated Stirling numbers of the second kind $S_{2,m}(n,k)$ reduce to the classical Stirling numbers of the second kind in (8).*

The truncated Stirling numbers of the second kind satisfy the following relationship.

Proposition 1. *The following correlation:*

$$S_{2,m}(n,k+l) = \frac{l!k!}{(k+l)!}\sum_{s=0}^{n}\binom{n}{s}S_{2,m}(s,k)\,S_{2,m}(n-s,l) \tag{22}$$

holds true for non-negative integers m and n.

Proof. In view of (8) and (21), we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} S_{2,m}(n,k+l)\frac{t^n}{n!} &= \frac{\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)^{k+l}}{(k+l)!} \\
&= \frac{l!k!}{(k+l)!}\frac{\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)^{k}}{k!}\frac{\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)^{l}}{l!} \\
&= \frac{l!k!}{(k+l)!}\sum_{n=0}^{\infty} S_{2,m}(n,k)\frac{t^n}{n!}\sum_{n=0}^{\infty} S_{2,m}(n,l)\frac{t^n}{n!} \\
&= \frac{l!k!}{(k+l)!}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s} S_{2,m}(s,k)\,S_{2,m}(n-s,l)\frac{t^n}{n!},
\end{aligned}$$

which gives the claimed result (22). □

We present the following correlation between two types of Stirling numbers of the second kind.

Proposition 2. *The following correlation:*

$$S_{2,1}(n,k) = 2^k S_2\left(n,k:\frac{1}{2}\right) \tag{23}$$

holds true for non-negative integers m and n.

Proof. In view of (8) and (21), we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} S_{2,1}(n,k)\frac{t^n}{n!} &= \frac{\left(e^t-1-1\right)^k}{k!} \\
&= \frac{2^k\left(\frac{1}{2}e^t-1\right)^k}{k!} \\
&= 2^k\sum_{n=0}^{\infty} S_2\left(n,k:\frac{1}{2}\right)\frac{t^n}{n!},
\end{aligned}$$

which presents the desired result (23). □

A relation that includes $F_{m,n}(x)$ and $S_{2,m}(n,k)$ is given by the following theorem.

Theorem 3. *The following relation:*

$$F_{m,n+m}(x) = \sum_{k=0}^{n}\binom{n+m}{m} x^k k! S_{2,m}(n,k) \tag{24}$$

is valid for a complex number x with $|x|<1$ and non-negative integers m and n.

Proof. By (16) and (21), we see that:

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{m,n}(x)\frac{t^n}{n!} &= \frac{\frac{t^m}{m!}}{1-x\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)} \\
&= \frac{t^m}{m!}\sum_{k=0}^{\infty} x^k\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)^k \\
&= \frac{t^m}{m!}\sum_{k=0}^{\infty} x^k k!\sum_{n=0}^{\infty} S_{2,m}(n,k)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x^k k! S_{2,m}(n,k)\frac{t^{n+m}}{m!n!},
\end{aligned}$$

which implies the desired result (24). □

We now state the following theorem.

Theorem 4. *The following identity:*

$$F_{1,n+1}(x) = n\sum_{k=1}^{\infty} x^k k! S_2\left(n,k:\frac{1}{2}\right) \tag{25}$$

holds true for a complex number x with $|x|<1$ and a positive integer n.

Proof. By (9) an (16), using the Cauchy product in series, we observe that:

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{1,n}(x)\frac{t^n}{n!} &= \frac{t}{1-x\left(e^t-2\right)} \\
&= t\sum_{k=0}^{\infty} x^k\left(e^t-2\right)^k \\
&= t\sum_{k=0}^{\infty} x^k k!\frac{\left(\frac{1}{2}e^t-1\right)^k}{k!}2^k \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x^k k! S_2\left(n,k:\frac{1}{2}\right)\frac{t^{n+1}}{n!},
\end{aligned}$$

which provides the asserted result (25). □

We now provide the derivative property for the polynomials $F_{m,n}(x,y)$ as follows.

Theorem 5. *The derivative formula:*

$$\frac{\partial}{\partial x}F_{m,n}(x,y) = nF_{m,n-1}(x,y) \tag{26}$$

holds true for non-negative integers m and a positive integer n.

Proof. Applying the derivative operator with respect to x to both sides of the equation (15), we acquire:

$$\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}\right)=\frac{\partial}{\partial x}\left(\frac{\frac{t^m}{m!}e^{xt}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\right)$$

and then:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}F_{m,n}\left(x,y\right)\frac{t^n}{n!} &= \frac{\frac{t^m}{m!}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\frac{\partial}{\partial x}e^{xt}\\ &= \frac{\frac{t^m}{m!}e^{xt}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}t\\ &= \sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^{n+1}}{n!},\end{aligned}$$

which means the claimed result (26). □

A recurrence relation for the two-variable truncated Fubini polynomials is given by the following theorem.

Theorem 6. *The following equalities:*

$$F_{m,n}\left(x,y\right)=0 \quad \left(n=0,1,2,\cdots,m-1\right)$$

and:

$$F_{m,n+m}\left(x,y\right)=\frac{y}{1+y}\sum_{s=0}^{n}\binom{n+m}{s}F_{m,s}\left(x,y\right)-\frac{x^n}{1+y}\frac{(n+m)!}{n!m!} \tag{27}$$

hold true for non-negative integers m and n.

Proof. Using Definition 1, we can write:

$$\begin{aligned}\frac{t^m}{m!}e^{xt} &= \left(1-y\left(\sum_{j=m}^{\infty}\frac{t^j}{j!}-1\right)\right)\sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}-y\left[\sum_{j=m}^{\infty}\frac{t^j}{j!}\sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}-\sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}\right]\\ &= \sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}-y\left[\sum_{j=0}^{\infty}\frac{t^{j+m}}{(j+m)!}\sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}-\sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}\right].\end{aligned}$$

Because of:

$$\sum_{j=0}^{\infty}\frac{t^{j+m}}{(j+m)!}\sum_{n=0}^{\infty}F_{m,n}\left(x,y\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n+m}{j}F_{m,j}\left(x,y\right)\frac{t^{n+m}}{(n+m)!},$$

we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty} x^n \frac{t^{n+m}}{n!m!} &= \sum_{n=0}^{\infty} F_{m,n}\left(x,y\right) \frac{t^n}{n!} - y \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n+m}{j} F_{m,j}\left(x,y\right) \frac{t^{n+m}}{(n+m)!} \\ &+ y \sum_{n=0}^{\infty} F_{m,n}\left(x,y\right) \frac{t^n}{n!}.\end{aligned}$$

Thus, we arrive at the following equality:

$$\sum_{n=0}^{\infty} F_{m,n}\left(x,y\right) \frac{t^n}{n!} = \frac{1}{1+y} \sum_{n=0}^{\infty} \left(y \sum_{j=0}^{n} \binom{n+m}{j} \frac{F_{m,j}\left(x,y\right)}{(n+m)!} - \frac{x^n}{n!m!} \right) t^{n+m}.$$

Comparing the coefficients of both sides of the last equality, the proof is completed. □

Theorem 6 can be used to determine the two-variable truncated Fubini polynomials. Thus, we provide some examples as follows.

Example 1. *Choosing $m = 1$, then we have $F_{1,0}\left(x,y\right) = 0$. Utilizing the recurrence formula (27), we derive:*

$$F_{1,n+1}\left(x,y\right) = \frac{y}{1-y} \sum_{s=0}^{n} \binom{n+1}{s} F_{1,s}\left(x,y\right) - \frac{x^n}{1-y}\left(n+1\right).$$

Thus, we subsequently acquire:

$$\begin{aligned}F_{1,1}\left(x,y\right) &= -\frac{1}{1+y}, \\ F_{1,2}\left(x,y\right) &= -\frac{2y}{(1+y)^2} - \frac{2x}{1+y}, \\ F_{1,3}\left(x,y\right) &= \frac{3}{1+y}\left(\frac{2y^2}{(1+y)^2} - \frac{2xy}{1+y} - x^2 \right).\end{aligned}$$

Furthermore, choosing $m = 2$, we then obtain the following recurrence relation:

$$F_{2,n+2}\left(x,y\right) = \frac{y}{1+y} \sum_{s=0}^{n} \binom{n+2}{s} F_{2,s}\left(x,y\right) - \frac{x^n}{1+y} \frac{(n+2)(n+1)}{2}$$

which yields the following polynomials:

$$\begin{aligned}F_{2,0}\left(x,y\right) &= F_{2,1}\left(x,y\right) = 0, \\ F_{2,2}\left(x,y\right) &= -\frac{1}{1+y}, \\ F_{2,3}\left(x,y\right) &= -\frac{3x}{1+y}, \\ F_{2,4}\left(x,y\right) &= \frac{6x}{y+1}\left(\frac{3y}{1+y} + x \right).\end{aligned}$$

By applying a similar method used above, one can derive the other two-variable truncated Fubini polynomials.

Here is a correlation that includes the truncated Fubini polynomials and Stirling numbers of the second kind.

Theorem 7. *For non-negative integers n and m, we have:*

$$F_{m,n}(x,y)=\sum_{u=0}^{n}\sum_{k=0}^{u}\binom{n}{u}F_{m,n-u}(y)\,S_2(u,k)\,(x)_k\,. \tag{28}$$

Proof. By means of Theorem 1 and Formula (8), we get:

$$\begin{aligned}F_{m,n}(x,y)&=\sum_{u=0}^{n}\binom{n}{u}F_{m,n-u}(y)\,x^u\\&=\sum_{u=0}^{n}\binom{n}{u}F_{m,n-u}(y)\sum_{k=0}^{u}S_2(u,k)\,(x)_k\,,\end{aligned}$$

which completes the proof of this theorem. □

The rising factorial number x is defined by $(x)^{(n)}=x(x+1)(x+2)\cdots(x+n-1)$ for a positive integer n. We also note that the negative binomial expansion is given as follows:

$$(x+a)^{-n}=\sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k a^{-n-k} \tag{29}$$

for negative integer $-n$ and $|x|<a$; cf. [7].

Here, we give the following theorem.

Theorem 8. *The following relationship:*

$$F_{m,n}(x,y)=\sum_{k=0}^{\infty}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\,F_{n,n-l}(-k,y)\,(x)^{(k)} \tag{30}$$

holds true for non-negative integers n and m.

Proof. By means of Definition 1 and using Equations (7) and (29), we attain:

$$\begin{aligned}\sum_{n=0}^{\infty}F_{m,n}(x,y)\frac{t^n}{n!}&=\frac{\frac{t^m}{m!}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\left(e^{-t}\right)^{-x}\\&=\frac{\frac{t^m}{m!}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\sum_{k=0}^{\infty}\binom{x+k-1}{k}\left(1-e^{-t}\right)^{-k}\\&=\frac{\frac{t^m}{m!}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\sum_{k=0}^{\infty}(x)^{(k)}\frac{\left(e^t-1\right)^k}{k!}e^{-kt}\\&=\sum_{k=0}^{\infty}(x)^{(k)}\sum_{n=0}^{\infty}F_{m,n}(-k,y)\frac{t^n}{n!}\sum_{n=0}^{\infty}S_2(n,k)\frac{t^n}{n!}\\&=\sum_{k=0}^{\infty}(x)^{(k)}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}F_{m,n-l}(-k,y)\,S_2(l,k)\right)\frac{t^n}{n!},\end{aligned}$$

which gives the asserted result (30). □

Therefore, we give the following theorem.

Theorem 9. *The following relationship:*

$$y\sum_{k=0}^{n}\binom{n}{k}F_{m,n-k}(z,y)\,F_{m+1,k}(x,y)=\sum_{k=0}^{n}\binom{n}{k}F_{m+1,n-k}(x,y)\,z^{k} \\ -\frac{n}{m+1}\sum_{k=0}^{n-1}\binom{n-1}{k}F_{m,n-1-k}(z,y)\,x^{k} \tag{31}$$

holds true for non-negative integers n and m.

Proof. By means of Definition 1, we see that:

$$\begin{aligned} e^{xt}\frac{t^{m+1}}{(m+1)!} &= \left(1-y\left(e^{t}-1-\sum_{j=0}^{m}\frac{t^{j}}{j!}\right)\right)\sum_{n=0}^{\infty}F_{m+1,n}(x,y)\frac{t^{n}}{n!} \\ &= \left(1-y\left(e^{t}-1-\sum_{j=0}^{m-1}\frac{t^{j}}{j!}\right)\right)\sum_{n=0}^{\infty}F_{m+1,n}(x,y)\frac{t^{n}}{n!} \\ &\quad -y\frac{t^{m}}{m!}\sum_{n=0}^{\infty}F_{m+1,n}(x,y)\frac{t^{n}}{n!}. \end{aligned}$$

Thus, we get:

$$e^{xt}\frac{t^{m+1}}{(m+1)!}\sum_{n=0}^{\infty}F_{m,n}(z,y)\frac{t^{n}}{n!}=\frac{t^{m}}{m!}e^{zt}\sum_{n=0}^{\infty}F_{m+1,n}(x,y)\frac{t^{n}}{n!} \\ -y\frac{t^{m}}{m!}\sum_{n=0}^{\infty}F_{m,n}(z,y)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}F_{m+1,n}(x,y)\frac{t^{n}}{n!}$$

and then:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}F_{m,n-k}(z,y)\,x^{k}\frac{t^{n+1}}{n!\,(m+1)}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}F_{m+1,n-k}(x,y)\,z^{k}\frac{t^{n}}{n!} \\ -y\sum_{n=0}^{\infty}y\sum_{k=0}^{n}\binom{n}{k}F_{m,n-k}(z,y)\,F_{m+1,k}(x,y)\frac{t^{n}}{n!}$$

which provides the claimed result in (31). □

Here, we investigate a linear combination for the two-variable truncated Fubini polynomials for different y values in the following theorem.

Theorem 10. *Let the numbers m and n be non-negative integers and $y_1 \neq y_2$. We then have:*

$$\frac{m!n!}{(n+m)!}\sum_{k=0}^{n+m}\binom{n+m}{k}F_{m,n+m-k}(x_1,y_1)\,F_{m,k}(x_2,y_2)=\frac{y_2F_{m,n-k}(x_1+x_2,y_2)-y_1F_{m,n-k}(x_1+x_2,y_1)}{y_2-y_1}. \tag{32}$$

Proof. By Definition 1, we consider the following product:

$$\frac{\frac{t^m}{m!}e^{x_1 t}}{1-y_1\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\frac{\frac{t^m}{m!}e^{x_2 t}}{1-y_2\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}$$
$$=\frac{y_2}{y_2-y_1}\frac{\frac{t^{2m}}{(m!)^2}e^{(x_1+x_2)t}}{1-y_2\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}-\frac{y_1}{y_2-y_1}\frac{\frac{t^{2m}}{(m!)^2}e^{(x_1+x_2)t}}{1-y_1\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)},$$

which yields

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}F_{m,n-k}\left(x_1,y_1\right)F_{m,k}\left(x_2,y_2\right)\frac{t^n}{n!}$$
$$=\frac{y_2}{y_2-y_1}\sum_{n=0}^{\infty}F_{m,n}\left(x_1+x_2,y_2\right)\frac{t^{n+m}}{n!m!}-\frac{y_1}{y_2-y_1}\sum_{n=0}^{\infty}F_{m,n}\left(x_1+x_2,y_1\right)\frac{t^{n+m}}{n!m!}.$$

Thus, we get:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{m,n-k}\left(x_1,y_1\right)F_{m,k}\left(x_2,y_2\right)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\frac{y_2}{y_2-y_1}F_{m,n}\left(x_1+x_2,y_2\right)-\frac{y_1}{y_2-y_1}F_{m,n}\left(x_1+x_2,y_1\right)\right)\frac{t^{n+m}}{n!m!},$$

which gives the desired result (32). □

3. Correlations with Truncated Euler and Bernoulli Polynomials

In this section, we investigate several correlations for the two-variable truncated Fubini polynomials $F_{m,n}(x,y)$ related to the truncated Euler polynomials $E_{m,n}(x)$ and numbers $E_{m,n}$ and the truncated Bernoulli polynomials $B_{m,n}(x)$ and numbers $B_{m,n}$.

Here is a relation between the truncated Euler polynomials and two-variable truncated Fubini polynomials at the special value $y=-\frac{1}{2}$.

Theorem 11. *We have:*

$$F_{m,n}\left(x,-\frac{1}{2}\right)=E_{m,n}\left(x\right). \tag{33}$$

Proof. In terms of (5) and (15), we get:

$$\begin{aligned}\sum_{n=0}^{\infty}F_{m,n}\left(x,-\frac{1}{2}\right)\frac{t^n}{n!} &= \frac{\frac{t^m}{m!}e^{xt}}{1+\frac{1}{2}\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\\ &= \frac{2\frac{t^m}{m!}e^{xt}}{e^t+1-\sum_{j=0}^{m-1}\frac{t^j}{j!}}\\ &= \sum_{n=0}^{\infty}E_{m,n}\left(x\right)\frac{t^n}{n!},\end{aligned}$$

which implies the asserted result (33). □

Corollary 1. *Taking $x = 0$, we then get a relation between the truncated Euler numbers and truncated Fubini polynomials at the special value $y = -\frac{1}{2}$, namely:*

$$F_{m,n}\left(-\frac{1}{2}\right) = E_{m,n}. \tag{34}$$

Remark 6. *The relations (33) and (34) are extensions of the relations in (12).*

We now state the following theorem, which includes a correlation for $F_{m,n}(x,y)$, $F_{m,n}(y)$ and $E_{m,n}(x)$.

Theorem 12. *The following formula:*

$$\begin{aligned} F_{m,n}(x,y) &= \frac{n!m!}{(n+m)!}\sum_{l=0}^{n+m}\frac{1}{2}\binom{n+m}{l}F_{m,l}(y)\,E_{m,n+m-l}(x) \\ &\quad+\frac{n!m!}{(n+m)!}\sum_{j=0}^{n}\frac{1}{2}\binom{n+m}{j}\sum_{l=0}^{j}\binom{j}{l}F_{m,l}(y)\,E_{m,j-l}(x) \end{aligned} \tag{35}$$

is valid for non-negative integers m and n.

Proof. By (5) and (15), we acquire that:

$$\begin{aligned} \sum_{n=0}^{\infty}F_{m,n}(x,y)\frac{t^n}{n!} &= \frac{\frac{t^m}{m!}e^{xt}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\frac{2\frac{t^m}{m!}}{e^t+1-\sum_{j=0}^{m-1}\frac{t^j}{j!}}\frac{e^t+1-\sum_{j=0}^{m-1}\frac{t^j}{j!}}{2\frac{t^m}{m!}} \\ &= \frac{1}{2}\frac{m!}{t^m}\sum_{n=0}^{\infty}F_{m,n}(y)\frac{t^n}{n!}\sum_{n=0}^{\infty}E_{m,n}(x)\frac{t^n}{n!}\left(\sum_{j=m}^{\infty}\frac{t^j}{j!}+1\right) \\ &= \frac{m!}{2}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}F_{m,l}(y)\,E_{m,n-l}(x)\right)\frac{t^{n-m}}{n!}\left(\sum_{j=0}^{\infty}\frac{t^{j+m}}{(j+m)!}+1\right) \\ &= \frac{m!}{2}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}F_{m,l}(y)\,E_{m,n-l}(x)\right)\frac{t^{n-m}}{n!} \\ &\quad+\frac{m!}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n+m}{j}\left(\sum_{l=0}^{j}\binom{j}{l}F_{m,l}(y)\,E_{m,j-l}(x)\right)\frac{t^n}{(n+m)!}, \end{aligned}$$

which completes the proof of the theorem. □

We finally state the relations for the truncated Bernoulli and Fubini polynomials as follows.

Theorem 13. *The following relation:*

$$F_{m,n}(x,y) = \frac{n!m!}{(n+m)!}\sum_{l=0}^{n}\binom{n+m}{l}\sum_{k=0}^{l}\binom{l}{k}F_{m,l}(y)\,B_{m,l-k}(x) \tag{36}$$

is valid for non-negative integers m and n.

Proof. By (5) and (15), we acquire that:

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{m,n}(x,y)\frac{t^n}{n!} &= \frac{\frac{t^m}{m!}e^{xt}}{1-y\left(e^t-1-\sum_{j=0}^{m-1}\frac{t^j}{j!}\right)}\frac{\frac{t^m}{m!}}{e^t-\sum_{j=0}^{m-1}\frac{t^j}{j!}}\frac{e^t-\sum_{j=0}^{m-1}\frac{t^j}{j!}}{\frac{t^m}{m!}} \\
&= \frac{m!}{t^m}\sum_{n=0}^{\infty} F_{m,n}(y)\frac{t^n}{n!}\sum_{n=0}^{\infty} B_{m,n}(x)\frac{t^n}{n!}\sum_{j=m}^{\infty}\frac{t^j}{j!} \\
&= m!\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{m,k}(y)B_{m,n-k}(x)\right)\frac{t^n}{n!}\sum_{j=0}^{\infty}\frac{t^j}{(j+m)!} \\
&= \frac{n!m!}{(n+m)!}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n+m}{l}\left(\sum_{k=0}^{l}\binom{l}{k}F_{m,l}(y)B_{m,l-k}(x)\right)\frac{t^n}{(n+m)!},
\end{aligned}$$

which means the asserted result (36). □

4. Conclusions

In this paper, we firstly considered two-variable truncated Fubini polynomials and numbers, and we then obtained some identities and properties for these polynomials and numbers, involving summation formulas, recurrence relations, and the derivative property. We also proved some formulas related to the truncated Stirling numbers of the second kind and Apostol-type Stirling numbers of the second kind. Furthermore, we gave some correlations including the two-variable truncated Fubini polynomials, the truncated Euler polynomials, and truncated Bernoulli polynomials.

Author Contributions: Both authors have equally contributed to this work. Both authors read and approved the final manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hassen, A.; Nguyen, H.D. Hypergeometric Bernoulli polynomials and Appell sequences. *Int. J. Number Theory* **2008**, *4*, 767–774. [CrossRef]
2. Kargın, L. Exponential polynomials and its applications to the related polynomials and numbers. *arXiv* **2016**, arXiv:1503.05444v2.
3. Kargın, L. Some formulae for products of Fubini polynomials with applications. *arXiv* **2016**, arXiv:1701.01023v1.
4. Khan, S.; Yasmin, G.; Ahmad, N. A note on truncated exponential-based Appell polynomials. *Bull. Malays. Math. Sci. Soc.* **2017**, *40*, 373–388. [CrossRef]
5. Kilar, N.; Simsek, Y. A new family of Fubini type numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials. *J. Korean Math. Soc.* **2017**, *54*, 1605–1621.
6. Kim, D.S.; Kim, T.; Kwon, H.-I.; Park, J.-W. Two variable higher-order Fubini polynomials. *J. Korean Math. Soc.* **2018**, *55*, 975–986.
7. Komatsu, T.; Ruiz, C.D.J.P. Truncated Euler polynomials. *Math. Slovaca* **2018**, *68*, 527–536. [CrossRef]
8. Luo, Q.-M.; Srivastava, H.M. Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind. *Appl. Math. Comput.* **2011**, *217*, 5702–5728. [CrossRef]
9. Srivastava, H.M.; Araci, S.; Khan, W.A.; Acikgoz, M. A note on the truncated-exponential based Apostol-type polynomials. *Symmetry* **2019**, *11*, 538. [CrossRef]
10. Su, D.D.; He, Y. Some identities for the two variable Fubini polynomials. *Mathematics* **2019**, *7*, 115. [CrossRef]

11. Dattoli, G.; Ceserano, C.; Sacchetti, D. A note on truncated polynomials. *Appl. Math. Comput.* **2003**, *134*, 595–605. [CrossRef]
12. Yasmin, G.; Khan, S.; Ahmad, N. Operational methods and truncated exponential-based Mittag-Leffler polynomials. *Mediterr. J. Math*. **2016**, *13*, 1555. [CrossRef]
13. Kim, T.; Kim, D.S.; Jang, G.-W. A note on degenerate Fubini polynomials. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 521–531.