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A New Class of Higher-Order Hypergeometric Bernoulli Polynomials Associated with Lagrange–Hermite Polynomials

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Abstract: The purpose of this paper is to construct a unified generating function involving the families of the higher-order hypergeometric Bernoulli polynomials and Lagrange–Hermite polynomials. Using the generating function and their functional equations, we investigate some properties of these polynomials. Moreover, we derive several connected formulas and relations including the Miller–Lee polynomials, the Laguerre polynomials, and the Lagrange Hermite–Miller–Lee polynomials.

Keywords: hypergeometric Bernoulli polynomials; Lagrange polynomials; hypergeometric Lagrange–Hermite–Bernoulli polynomials; confluent hypergeometric function; special polynomials

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1. Introduction

Special polynomials (like Bernoulli, Euler, Hermite, Laguerre, etc.) have great importance in applied mathematics, mathematical physics, quantum mechanics, engineering, and other fields of mathematics. Particularly the family of special polynomials is one of the most useful, widespread, and applicable families of special functions. Recently, the aforementioned polynomials and their diverse extensions have been studied and introduced in [1–14].

In this paper, the usual notations refer to the set of all complex numbers $\mathbb{C}$, the set of real numbers $\mathbb{R}$, the set of all integers $\mathbb{Z}$, the set of all natural numbers $\mathbb{N}$, and the set of all non-negative integers $\mathbb{N}_0$, respectively. The classical Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad (|t|<2\pi). \tag{1}$$

Upon setting $x=0$ in (1), the Bernoulli polynomials reduce to the Bernoulli numbers, namely, $B_n(0):=B_n$. The Bernoulli numbers and polynomials have a long history, which arise from Bernoulli calculations of power sums in 1713 (see [9]), that is

$$\sum_{j=1}^{m}j^n=\frac{B_{n+1}(m+1)-B_{n+1}}{n+1}$$

The Bernoulli polynomials have many applications in modern number theory, such as modular forms and Iwasawa theory [11].

In 1924, Nörlund [13] introduced the Bernoulli polynomials and numbers of order α :

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{zt}=\frac{e^{zt}}{\left(\frac{e^t-1}{t}\right)^{\alpha}}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(z)\frac{t^n}{n!}. \tag{2}$$

For $M, N \in \mathbb{N}$, and $\alpha \in \mathbb{C}$, Su and Komatsu [10] defined the hypergeometric Bernoulli polynomials $B^{(\alpha)}_{M,N,n}(x)$ of order α by means of the following generating function:

$$\frac{e^{xt}}{{}_1F_1(M;M+N;t)^{\alpha}} = \sum_{n=0}^{\infty} B^{(\alpha)}_{M,N,n}(x)\frac{t^n}{n!}, \tag{3}$$

where

$${}_1F_1(M;M+N;t) = \sum_{n=0}^{\infty} \frac{(M)_n}{(M+N)_n}\frac{t^n}{n!}$$

is called the confluent hypergeometric function (see [14]) with $(x)_n := x(x+1)\cdots(x+n-1)$ for $n \in \mathbb{N}$ and $(x)_0 = 1$. When $x = 0$, $B^{(\alpha)}_{M,N,n}(0) := B^{(\alpha)}_{M,N,n}$ are the higher-order generalized hypergeometric Bernoulli numbers. When $M = 1$, the higher-order hypergeometric Bernoulli polynomials $B^{(\alpha)}_{N,n}(x) := B^{(\alpha)}_{1,N,n}(x)$, which are studied by Hu and Kim in [9]. When $\alpha = M = 1$, we have that $B_{N,n}(x) = B_{N,n}(x)$ are the hypergeometric Bernoulli polynomials which are defined by Howard [7,8] as

$$\frac{e^{xt}}{{}_1F_1(1;1+N;t)} = \frac{t^N e^{xt}/N!}{e^t - T_{N-1}(t)} = \sum_{n=0}^{\infty} B_{N,n}(x)\frac{t^n}{n!}. \tag{4}$$

For $\alpha = M = N = 1$ in (3), we have $B^{(1)}_{1,1,n}(x) := B_n(x)$.

The Lagrange polynomials in several variables, which are known as the Chan–Chyan–Srivastava polynomials [2], are defined by means of the following generating function:

$$\prod_{j=1}^{r}(1-x_j t)^{-\alpha_j} = \sum_{n=0}^{\infty} g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)t^n, \tag{5}$$
$$\Big(\alpha_j \in \mathbb{C} \quad (j=1,\cdots,r); \;\; |t| < \min\{|x_1|^{-1},\cdots,|x_r|^{-1}\}\Big),$$

and are represented by

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{k_1+\cdots+k_r=n} (\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}. \tag{6}$$

Altin and Erkus [1] introduced the multivariable Lagrange–Hermite polynomials given by

$$\prod_{j=1}^{r}(1-x_j t^j)^{-\alpha_j} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots x_r)t^n, \tag{7}$$
$$(\alpha_j \in \mathbb{C} \quad (j=1,\cdots,r)); |t| < \min\{|x_1|^{-1},|x_2|^{-\frac{1}{2}},\cdots,|x_r|^{-\frac{1}{r}}\},$$

where

$$h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{k_1+2k_2+\cdots+rk_r=n} (\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}.$$

In the special case when $r = 2$ in (7), the polynomials $h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots x_r)$ reduce to the familiar (two-variable) Lagrange–Hermite polynomials considered by Dattoli et al. [3]:

$$(1-x_1 t)^{-\alpha_1}(1-x_2 t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)t^n. \tag{8}$$

The multivariable (Erkus–Srivastava) polynomials $U_{n;l_1,\cdots,l_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ are defined by the following generating function [6]:

$$\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j}=\sum_{n=0}^{\infty}U_{n;l_1,\cdots,l_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)t^n, \tag{9}$$

$$(\alpha_j\in\mathbb{C},\, l_j\in\mathbb{N}\;\; (j=1,\cdots,r);\; |t|<\min\{|x_1|^{-1/l_1},\cdots,|x_r|^{-1/l_r}\})$$

which are a unification (and generalization) of several known families of multivariable polynomials including the Chan–Chyan–Srivastava polynomials $g_n^{(\alpha_1\cdots\alpha_r)}(x_1,\cdots,x_r)$ in (5) and multivariable Lagrange–Hermite polynomials (7).

By (9), the Erkus–Srivastava polynomials $U_{n,l_1,\cdots,l_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ satisfy the following explicit representation (cf. [6]):

$$U_{n;l_1,\cdots,l_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)=\sum_{l_1k_1+\cdots+l_rk_r=n}(\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}, \tag{10}$$

which is the generalization of Relation (6).

In this paper, we introduce the multivariable unified Lagrange–Hermite-based hypergeometric Bernoulli polynomials and investigate some of their properties. Then, we derive multifarious connected formulas involving the Miller–Lee polynomials, the Laguerre polynomials polynomials, the Lagrange Hermite–Miller–Lee polynomials.

2. Lagrange–Hermite-Based Hypergeometric Bernoulli Polynomials

By means of (3) and (9), we consider a unification of the hypergeometric Bernoulli polynomials $B_{M,N,n}^{(\alpha)}(x)$ of order α and the multivariable (Erkus–Srivastava) polynomials $U_{n,l_1,\cdots,l_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$. Thus, we define the multivariable unified Lagrange–Hermite-based hypergeometric Bernoulli polynomials ${}_HB_{M,N,n;l_1,\cdots,l_r}^{(\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r)$ of order $\alpha\in\mathbb{C}$ by means of the following generating function:

$$\frac{1}{({}_1F_1(M;M+N;t))^{\alpha}}e^{xt}\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j}=\sum_{n=0}^{\infty}{}_HB_{M,N,n;l_1,\cdots,l_r}^{(\alpha|\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r)t^n, \tag{11}$$

where $\alpha_j\in\mathbb{C}$, $l_j\in\mathbb{N}$ for $j=1,\cdots,r$ and $|t|<\min\{|x_1|^{-1/l_1},\cdots,|x_r|^{-1/l_r}\}$. Upon setting $l_j=j$, we have ${}_HB_{M,N,n;1,\cdots,r}^{(\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r):={}_HB_{M,N,n}^{(\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r)$, which we call the multivariable Lagrange–Hermite-based hypergeometric Bernoulli polynomials of order α :

$$\frac{1}{({}_1F_1(M;M+N;t))^{\alpha}}e^{xt}\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j}=\sum_{n=0}^{\infty}{}_HB_{M,N,n}^{(\alpha|\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r)t^n \tag{12}$$

where $\alpha_j\in\mathbb{C}$ for $j=1,\cdots,r$ and $|t|<\min\{|x_1|^{-1},\cdots,|x_r|^{-1/r}\}$. Furthermore, note that

$${}_HB_{M,N,n;l_1,\cdots,l_r}^{(1|\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r):={}_HB_{M,N,n;l_1,\cdots,l_r}^{(\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r).$$

Remark 1. *In the case $l_j=j$ and $r=2$, we get ${}_HB_{M,N,n;1,\cdots,r}^{(\alpha|\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r):={}_HB_{M,N,n}^{(\alpha|\alpha_1,\cdots,\alpha_r)}$ $(x|x_1,x_2)$, which we call the Lagrange–Hermite-based hypergeometric Bernoulli polynomials of order α:*

$$\frac{e^{xt}}{({}_1F_1(M;M+N;t))^{\alpha}}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}=\sum_{n=0}^{\infty}{}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n. \tag{13}$$

Remark 2. *When $l_j = 1$ and $r = 2$, we acquire ${}_HB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n;1,\cdots,1}(x|x_1,\cdots,x_r) := {}_gB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n}(x|x_1,x_2)$, which we call the Lagrange-based hypergeometric Bernoulli polynomials of order α, and which are defined by*

$$\frac{e^{xt}}{{}_1F_1(M;M+N;t)^{\alpha}}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_gB^{(\alpha|\alpha_1,\alpha_2)}_{M,N,n}(x|x_1,x_2)t^n. \tag{14}$$

When $x = 0$ in (14), we have ${}_gB^{(\alpha|\alpha_1,\alpha_2)}_{M,N,n}(0|x_1,x_2) = {}_gB^{(\alpha|\alpha_1,\alpha_2)}_{M,N,n}(x_1,x_2)$, which we call the Lagrange-based hypergeometric Bernoulli numbers of order α.

We now investigate some properties of ${}_HB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n;l_1,\cdots,l_r}(x|x_1,\cdots,x_r)$.

Theorem 1. *The following summation formula:*

$${}_HB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n;l_1,\cdots,l_r}(x|x_1,\cdots,x_r) = \sum_{s=0}^{n} U^{(\alpha_1,\cdots,\alpha_r)}_{n-s;l_1,\cdots,l_r}(x_1,\cdots,x_r)\frac{B^{(\alpha)}_{M,N,s}(x)}{s!} \tag{15}$$

holds for $n \in \mathbb{N}_0$.

Proof. By (11), we have

$$\sum_{n=0}^{\infty} {}_HB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n;l_1,\cdots,l_r}(x|x_1,\cdots,x_r)t^n = \frac{e^{xt}}{({}_1F_1(M;M+N;t))^{\alpha}}\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j}$$

$$= \sum_{n=0}^{\infty} B^{(\alpha)}_{M,N,n}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} U^{(\alpha_1,\cdots,\alpha_r)}_{n,l_1,\cdots,l_r}(x_1,\cdots,x_r)t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{n} U^{(\alpha_1,\cdots,\alpha_r)}_{n-s;l_1,\cdots,l_r}(x_1,\cdots,x_r)\frac{B^{(\alpha)}_{M,N,s}(x)}{s!}t^n,$$

which gives the asserted Formula (15). □

Theorem 2. *The following summation formula:*

$${}_HB^{(\alpha+\beta|\alpha_1,\cdots,\alpha_r)}_{M,N,n;l_1,\cdots,l_r}(x+y|x_1,\cdots,x_r) = \sum_{m=0}^{n} {}_HB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n-m;l_1,\cdots,l_r}(x|x_1,\cdots,x_r)\frac{B^{(\beta)}_{M,N,m}(y)}{m!} \tag{16}$$

holds for $n \in \mathbb{N}_0$.

Proof. By using (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HB^{(\alpha+\beta|\alpha_1,\cdots,\alpha_r)}_{M,N,n;l_1,\cdots,l_r}(x+y|x_1,\cdots,x_r)t^n &= \frac{e^{xt}}{({}_1F_1(M;M+N;t))^{\alpha}}\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j}\frac{e^{yt}}{({}_1F_1(M;M+N;t))^{\beta}} \\ &= \sum_{n=0}^{\infty} {}_HB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n;l_1,\cdots,l_r}(x|x_1,\cdots,x_r)t^n\sum_{m=0}^{\infty} B^{(\beta)}_{M,N,m}(y)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n-m;l_1,\cdots,l_r}(x|x_1,\cdots,x_r)B^{(\beta)}_{M,N,m}(y)\frac{t^n}{m!},\end{aligned}$$

which gives the asserted result (16). □

We give the following theorem:

Theorem 3. *The following summation formula:*

$${}_HB^{(\alpha|\alpha_1-\beta_1,\cdots,\alpha_r-\beta_r)}_{M,N,n;l_1,\cdots,l_r}(x|x_1,\cdots,x_r) = \sum_{m=0}^{n} {}_HB^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n-m;l_1,\cdots,l_r}(x|x_1,\cdots,x_r)U^{(-\beta_1,\cdots,-\beta_r)}_{m;l_1,\cdots,l_r}(x_1,\cdots,x_r) \tag{17}$$

holds for $n \in \mathbb{N}_0$.

Proof. Using definition (11), we have

$$\sum_{n=0}^{\infty} {}_H B^{(\alpha|\alpha_1-\beta_1,\cdots,\alpha_r-\beta_r)}_{M,N,n;l_1,\cdots,l_r}(x|x_1,\cdots,x_r)t^n = \frac{e^{xt}}{({}_1F_1(M;M+N;t))^{\alpha}} \prod_{j=1}^{r}(1-x_j t^{l_j})^{\beta_j-\alpha_j}$$
$$= \frac{e^{xt}}{({}_1F_1(M;M+N;t))^{\alpha}} \prod_{j=1}^{r}(1-x_j t^{l_j})^{-\alpha_j} \prod_{j=1}^{r}(1-x_j t^{l_j})^{\beta_j}$$
$$= \sum_{n=0}^{\infty} {}_H B^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n;l_1,\cdots,l_r}(x|x_1,\cdots,x_r)t^n \sum_{n=0}^{\infty} U^{(-\beta_1,\cdots,-\beta_r)}_{n;l_1,\cdots,l_r}(x_1,\cdots,x_r)t^n,$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_H B^{(\alpha|\alpha_1,\cdots,\alpha_r)}_{M,N,n-m;l_1,\cdots,l_r}(x|x_1,\cdots,x_r) U^{(-\beta_1,\cdots,-\beta_r)}_{m;l_1,\cdots,l_r}(x_1,\cdots,x_r)t^n,$$

which provides the claimed result (17). □

We state the following theorem:

Theorem 4. *The following summation formulas for the higher-order generalized hypergeometric Lagrange–Hermite–Bernoulli polynomials* ${}_H B^{(\alpha|\alpha_1,\alpha_2)}_{M,N,n}(x|x_1,x_2)$ *hold:*

$$\int_0^1 x^{M-1}(1-x)^{N-1} {}_H B^{(1|0,0)}_{M,N,n}(x|1,1)dx = \frac{\Gamma(N)}{\Gamma(n+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(M+n-k)}{\Gamma(M+N+n-k)} B_k(M,N), \tag{18}$$

and

$$x^n = \frac{\Gamma(M+N)}{\Gamma(M)} \sum_{k=0}^{n} \frac{\Gamma(M+n-k)}{\Gamma(M+N+n-k)} {}_H B^{(1|0,0)}_{M,N,k}(x|1,1) \frac{n!}{(n-k)!}. \tag{19}$$

Proof. For $\alpha = 1$ and $\alpha_1 = \alpha_2 = 0$ in (13), we have

$$\frac{1}{{}_1F_1(M;M+N;t)} e^{xt} = \sum_{n=0}^{\infty} {}_H B^{(1|0,0)}_{M,N,n}(x|1,1)t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} B_k(M,N) x^{n-k} \frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_H B^{(1|0,0)}_{M,N,n}(x|1,1)t^n. \tag{20}$$

Moreover, we have

$$x^n = \frac{\Gamma(M+N)}{\Gamma(M)} \sum_{k=0}^{n} \frac{\Gamma(M+n-k)}{\Gamma(M+N+n-k)} {}_H B^{(1|0,0)}_{M,N,k}(x|1,1) \frac{n!}{(n-k)!}.$$

Therefore, by integrating (20) with weight $(1-x)^{N-1}x^{M-1}$, we obtain

$$\int_0^1 x^{M-1}(1-x)^{N-1} {}_H B^{(1|0,0)}_{M,N,n}(x|1,1)dx$$
$$= \sum_{k=0}^{n} \binom{n}{k} B_k(M,N) \frac{1}{n!} \int_0^1 x^{M+n-k-1}(1-x)^{N-1}dx$$
$$= \frac{\Gamma(N)}{\Gamma(n+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(M+n-k)}{\Gamma(M+N+n-k)} B_k(M,N),$$

which completes the proof. □

Theorem 5. *The following summation formula for the higher-order generalized hypergeometric Lagrange–Hermite–Bernoulli polynomials* ${}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *holds:*

$$h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)=\frac{\Gamma(M+N)}{\Gamma(M)}\sum_{k=0}^{n}\frac{\Gamma(M+n-k)}{\Gamma(M+N+n-k)}\,{}_HB_{M,N,k}^{(1|\alpha_1,\alpha_2)}(0|x_1,x_2)\frac{1}{(n-k)!}. \tag{21}$$

Proof. For $\alpha=1$ and $x=0$ in (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty}h_n^{(\alpha_1,\alpha_2)}(x_1,x_2)t^n &= (1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}={}_1F_1(M;M+N;t)\sum_{n=0}^{\infty}{}_HB_{M,N,n}^{(1|\alpha_1,\alpha_2)}(0|x_1,x_2)t^n\\ &=\sum_{n=0}^{\infty}\frac{(M)_n}{(M+N)_n}\frac{t^n}{n!}\sum_{k=0}^{\infty}{}_HB_{M,N,k}^{(1|\alpha_1,\alpha_2)}(0|x_1,x_2)t^k\\ &=\frac{\Gamma(M+N)}{\Gamma(M)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\Gamma(M+n-k)}{\Gamma(M+N+n-k)}\,{}_HB_{M,N,k}^{(1|\alpha_1,\alpha_2)}(0|x_1,x_2)\frac{t^n}{(n-k)!}.\end{aligned}$$

Comparing the coefficients of t^n in both sides, we get the result (21). □

We give the following derivative property:

Theorem 6. *The following derivative property for the higher-order hypergeometric generalized Lagrange–Hermite–Bernoulli polynomials* ${}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *holds:*

$$\frac{d^p}{dx^p}\,{}_HB_{M,N,n;l_1,\cdots,l_r}^{(\alpha|\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r)={}_HB_{M,N,n-p;l_1,\cdots,l_r}^{(\alpha|\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r),\quad n\geq p. \tag{22}$$

Proof. Start with

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d^p}{dx^p}\,{}_HB_{M,N,n;l_1,\cdots,l_r}^{(\alpha|\alpha_1,\cdots,\alpha_r)}(x|x_1,\cdots,x_r)t^n &= \frac{\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j}}{({}_1F_1(M;M+N;t))^{\alpha}}\frac{d^p}{dx^p}e^{xt}\\ &= \frac{\prod_{j=1}^{r}(1-x_jt^{l_j})^{-\alpha_j}}{({}_1F_1(M;M+N;t))^{\alpha}}e^{xt}t^p\\ &= \sum_{n=0}^{\infty}{}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)t^{n+p},\end{aligned}$$

which implies the asserted result (22). □

Theorem 7. *The following summation formula involving the higher-order generalized hypergeometric Lagrange–Hermite–Bernoulli polynomials* ${}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and higher-order generalized hypergeometric Lagrange–Bernoulli polynomials* ${}_gB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *holds true:*

$$\sum_{m=0}^{n}{}_HB_{M,N,n-m}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{(\beta)_my^m}{m!}=\sum_{m=0}^{n}{}_gB_{M,N,n-m}^{(\alpha|\alpha_1,\beta)}(x|x_1,y)\frac{(x_2)^m}{m!}(\alpha_2)_m. \tag{23}$$

Proof. The proof is similar to Theorem 3. □

3. Some Connected Formulas

The generation functions (13) and (14) can be exploited in a number of ways and provide a useful tool to frame known and new generating functions in the following way: As a first example, we set $\alpha=\alpha_2=0$, $\alpha_1=m+1$, $x_1=1$ in (13) to get

$$e^{xt}(1-t)^{-m-1}=\sum_{n=0}^{\infty}G_n^{(m)}(x)t^n,\quad |t|<1, \tag{24}$$

where $G_n^{(m)}(x)$ are called the Miller–Lee polynomials (see [4]).

Another example is the definition of higher-order hypergeometric Bernoulli–Hermite–Miller–Lee polynomials ${}_HB_{M,N,n}^{(m,\alpha)}(x,y)$ given by the following generating function:

$$\frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}\frac{e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}}{(1-t)^{m+1}}=\sum_{n=0}^{\infty}{}_BHG_{M,N,n}^{(m,\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{n!},\quad (25)$$

which for $\alpha=0$ reduces to

$$\frac{e^{xt}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2}}{(1-t)^{m+1}}=\sum_{n=0}^{\infty}{}_HG_n^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{n!},\quad (26)$$

where ${}_HG_n^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)$ are called the Lagrange Hermite–Miller–Lee polynomials.

Putting $\alpha_1=\alpha_2=0$ into (25) gives

$$\frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}\frac{e^{xt}}{(1-t)^{m+1}}=\sum_{n=0}^{\infty}{}_BG_{M,N,n}^{(m,\alpha)}(x)\frac{t^n}{n!},\quad (27)$$

where ${}_BG_{M,N,n}^{(m,\alpha)}(x)$ are called the higher-order hypergeometric Bernoulli–Miller–Lee polynomials.

We now give some connected formulas as follows:

Theorem 8. *The following implicit summation formula involving higher-order hypergeometric Lagrange–Hermite–Bernoulli polynomials ${}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$, Bernoulli–Miller–Lee polynomials ${}_BG_{M,N,n}^{(m,\alpha)}(x)$ and Miller–Lee polynomials $G_n^{(m)}(x)$ holds:*

$${}_BG_{M,N,n}^{(m,\alpha)}(x)=n!\sum_{r=0}^{n}B_{M,N,n-r}^{(\alpha)}G_r^{(m)}(x)\frac{1}{(n-r)!}=n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(-\alpha_2)_r(x_2)^r}{r!}{}_HB_{M,N,n-2r}^{(\alpha|m+1,\alpha_2)}(x|1,x_2).\quad (28)$$

Proof. For $x_1=1$ and $\alpha_1=m+1$ in (13) and using (27), we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_BG_{M,N,n}^{(m,\alpha)}(x)\frac{t^n}{n!}&=\frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{xt}(1-t)^{-m-1}\\&=(1-x_2t^2)^{\alpha_2}\sum_{n=0}^{\infty}{}_HB_{M,N,n}^{(\alpha|m+1,\alpha_2)}(x|1,x_2)t^n\end{aligned}$$

which by using binomial expansion takes the form

$$\begin{aligned}\sum_{n=0}^{\infty}B_{M,N,n}^{(\alpha)}\frac{t^n}{n!}\sum_{r=0}^{\infty}G_r^{(m)}(x)t^r&=\sum_{r=0}^{\infty}\frac{(-\alpha_2)_r(x_2)^rt^{2r}}{r!}\sum_{n=0}^{\infty}{}_HB_{M,N,n}^{(\alpha|m+1,\alpha_2)}(x|1,x_2)t^n\\&\sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]}\frac{(-\alpha_2)_r(x_2)^r}{r!}{}_HB_{M,N,n-2r}^{(\alpha|m+1,\alpha_2)}(x|1,x_2)t^n,\end{aligned}$$

which implies the asserted result (28). □

Theorem 9. *The following implicit summation formula involving higher-order Lagrange–Hermite–Bernoulli polynomials ${}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ and Miller–Lee polynomials $G_n^{(m)}(x)$ holds:*

$${}_HB_{M,N,n}^{(\alpha|\alpha_1+m+1,\alpha_2)}(x+y|x_1,x_2)=\sum_{r=0}^{n}{}_HB_{M,N,n-r}^{(\alpha|\alpha_1,\alpha_2)}(y|x_1,x_2)G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r.\quad (29)$$

Proof. On replacing x with $x+y$ and α_1 with α_1+m+1, respectively, in (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HB^{(\alpha|\alpha_1+m+1,\alpha_2)}_{M,N,n}(x+y|x_1,x_2)t^n &= \frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{(x+y)t} \\ &\quad\times(1-x_1t)^{-m-1}(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} \\ &= \sum_{n=0}^{\infty} {}_HB^{(\alpha|\alpha_1,\alpha_2)}_{M,N,n}(y|x_1,x_2)t^n \sum_{r=0}^{\infty} G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r t^r \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_HB^{(\alpha|\alpha_1,\alpha_2)}_{M,N,n-r}(y|x_1,x_2)G_r^{(m)}\left(\frac{x}{x_1}\right)x_1^r t^n,\end{aligned}$$

which yields the claimed result (29). □

Theorem 10. *The following implicit summation formula involving higher-order Lagrange–Hermite–Bernoulli polynomials* ${}_HB^{(\alpha|\alpha_1,\alpha_2)}_{M,N,n}(x|x_1,x_2)$ *and Miller–Lee polynomials* $G_n^{(m)}(x)$ *holds:*

$$\sum_{r=0}^{n} B^{(\alpha)}_{M,N,n-r}\,{}_HG_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{1}{(n-r)!} = \sum_{r=0}^{n}(\alpha_1)_r x_1^r\,{}_HB^{(\alpha|m+1,\alpha_2)}_{M,N,n-r}(x|1,x_2)\frac{1}{r!}. \tag{30}$$

Proof. For $\alpha_1 = m+1$ and $x_1 = 1$ in (13), we have

$$\sum_{n=0}^{\infty} {}_HB^{(\alpha|m+1,\alpha_2)}_{M,N,n}(x|1,x_2)t^n = \frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{xt}(1-t)^{-m-1}(1-x_2t^2)^{-\alpha_2}.$$

Multiplying both the sides by $(1-x_1t)^{-\alpha_1}$, we have

$$\sum_{n=0}^{\infty} B^{(\alpha)}_{M,N,n}\frac{t^n}{n!}\sum_{r=0}^{\infty} {}_HG_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)t^n = \sum_{n=0}^{\infty}\sum_{r=0}^{n} B^{(\alpha)}_{M,N,n-r}\,{}_HG_r^{(m|\alpha_1,\alpha_2)}(x|x_1,x_2)\frac{t^n}{(n-r)!}.$$

Now, replacing n by $n-r$ in the above equation, we get

$$(1-x_1t)^{-\alpha_1}\sum_{n=0}^{\infty} {}_HB^{(\alpha|m+1,\alpha_2)}_{M,N,n}(x|1,x_2)t^n = \sum_{n=0}^{\infty}\sum_{r=0}^{n}(\alpha_1)_r(x_1)^r\,{}_HB^{(\alpha|m+1,\alpha_2)}_{M,N,n-r}(x|1,x_2)\frac{t^n}{r!}.$$

Comparing the coefficient of t^n, we get the result (30). □

Now, we shall focus on the connection between the higher-order generalized hypergeometric Lagrange–Hermite–Bernoulli polynomials ${}_HB^{(\alpha|\alpha_1,\alpha_2)}_{M,N,n}(x|x_1,x_2)$ and Laguerre polynomials $L_n^{(m)}(x)$.

For $x_2 = 0$, $x_1 = -1$, $\alpha_1 = -m$ and $\alpha_2 = 0$ in Equation (11), we have

$$\frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{xt}(1+t)^m = \sum_{n=0}^{\infty} {}_BL^{(\alpha|m)}_{M,N,n}(x)\frac{t^n}{n!}, \tag{31}$$

where ${}_HB^{(\alpha|-m,0)}_{M,N,n}(x|-1,0) = {}_BL^{(\alpha|m)}_{M,N,n}(x)$ are called generalized higher-order hypergeometric Bernoulli–Laguerre polynomials.

When $\alpha = 0$ in (31), ${}_BL^{(\alpha|m)}_{M,N,n}(x)$ reduces to ordinary Laguerre polynomials $L_n^{(m)}(x)$ (see [14]).

Theorem 11. *The following implicit summation formula involving higher-order Lagrange–Hermite–Bernoulli polynomials* ${}_{H}B_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and Laguerre polynomials* $L_n^{(m)}(x)$ *holds:*

$$\sum_{r=0}^{n} {}_{H}B_{M,N,n-r}^{(\alpha)}(x|x_1,x_2)L_r^{(m-r)}(y) = \sum_{r=0}^{n}(\alpha)_r(x_1)^r {}_{H}B_{M,N,n-r}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)\frac{1}{r!}. \tag{32}$$

Proof. By replacing x with $x+y$ and setting $x_1=-1$, $\alpha_1=-m$ in (13), we have

$$\frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{(x+y)t}(1+t)^m(1-x_2t^2)^{-\alpha_2} = \sum_{n=0}^{\infty} {}_{H}B_{M,N,n}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)t^n.$$

Multiplying both sides $(1-x_1t)^{-\alpha_1}$, we have

$$\begin{aligned} & \frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{(x+y)t}(1+t)^m(1-x_1t)^{-\alpha_1}(1-x_2t^2)^{-\alpha_2} \\ = & (1-x_1t)^{-\alpha_1}\sum_{n=0}^{\infty} {}_{H}B_{M,N,n}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)t^n \\ = & \sum_{n=0}^{\infty} {}_{H}B_{M,N,n}^{(\alpha)}(x|x_1,x_2)t^n\sum_{r=0}^{\infty}L_r^{(m-r)}(y)t^r \\ = & \sum_{r=0}^{\infty}\frac{(\alpha)_r(x_1)^rt^r}{r!}\sum_{n=0}^{\infty} {}_{H}B_{M,N,n}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)t^n, \end{aligned}$$

which gives

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_{H}B_{M,N,n-r}^{(\alpha)}(x|x_1,x_2)L_r^{(m-r)}(y)t^n = \sum_{n=0}^{\infty}\sum_{r=0}^{n}(\alpha)_r(x_1)^r {}_{H}B_{M,N,n-r}^{(\alpha|-m,\alpha_2)}(x+y|-1,x_2)\frac{t^n}{r!},$$

which yields the asserted result (32). □

Theorem 12. *The following implicit summation formula involving higher-order hypergeometric Lagrange–Hermite–Bernoulli polynomials* ${}_{B}H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and Laguerre polynomials* $L_n^{(m)}(x)$ *holds true:*

$$\sum_{k=0}^{n} B_{M,N,n-k}^{(\alpha)}(x)L_k^{(m-k)}(y)\frac{1}{(n-k)!} = {}_{H}B_{M,N,n}^{(\alpha|-m,0)}(x+y|-1,x_2). \tag{33}$$

Proof. By replacing x with $x+y$ and setting $x_1=-1$, $\alpha_1=-m$, and $\alpha_2=0$ in Equation (11), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{H}B_{M,N,n}^{(\alpha|-m,0)}(x+y|-1,x_2)t^n &= \frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{(x+y)t}(1+t)^m \\ &= \sum_{n=0}^{\infty}B_{M,N,n}^{(\alpha)}(x)\frac{t^n}{n!}\sum_{k=0}^{\infty}L_k^{(m-k)}(y)t^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}B_{M,N,n-k}^{(\alpha)}(x)L_k^{(m-k)}(y)\frac{t^n}{(n-k)!} \end{aligned}$$

which yields the asserted result (33). □

Theorem 13. *The following implicit summation formula involving the Lagrange–Hermite–Bernoulli polynomials* ${}_{B}H_n^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and Laguerre polynomials* $L_n^{(m)}(x)$ *holds true:*

$$\sum_{k=0}^{n} {}_{H}B_{M,N,n-k}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2)(-x_1)^kL_k^{(m-k)}(y/x_1) = {}_{H}B_{M,N,n}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2). \tag{34}$$

Proof. Replacing α_1 with $-m+\alpha_1$ and $x \longrightarrow x-y$ in (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HB_{M,N,n}^{(\alpha|-m+\alpha_1,\alpha_2)}(x-y|x_1,x_2)t^n &= \frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{(x-y)t}(1-x_1t)^{m-\alpha_1}(1-x_2t^2)^{-\alpha_2} \\ &= \sum_{n=0}^{\infty} {}_HB_{M,N,n}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2)t^n \sum_{k=0}^{\infty}(-x_1)^k t^k L_k^{(m-k)}(y/x_1) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_HB_{M,N,n-k}^{(\alpha|-m+\alpha_1,\alpha_2)}(x|x_1,x_2)(-x_1)^k L_k^{(m-k)}(y/x_1)t^n,\end{aligned}$$

which implies the claimed result (34). □

Theorem 14. *The following implicit summation formula involving higher-order Lagrange–Hermite–Bernoulli polynomials* ${}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and Laguerre polynomials* $L_n^{(m)}(x)$ *holds:*

$$\sum_{k=0}^{n} B_{M,N,n-k}^{(\alpha)}(x)L_k^{(m-k)}(y)\frac{1}{(n-k)!} = {}_HB_{M,N,n}^{(\alpha|-m,0)}(x+y|-1,x_2). \tag{35}$$

Proof. For $x_1=-1$, $\alpha_1=-m$, $\alpha_2=0$ and replacing x with $x-y$ in (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HB_{M,N,n}^{(\alpha|-m,0)}(x-y|-1,x_2)t^n &= \frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{(x-y)t}(1+t)^m \\ &= \sum_{n=0}^{\infty} B_{M,N,n}^{(\alpha)}(x)\frac{t^n}{n!}\sum_{k=0}^{\infty} L_k^{(m-k)}(-y)t^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{M,N,n-k}^{(\alpha)}(x)L_k^{(m-k)}(-y)\frac{t^n}{(n-k)!},\end{aligned}$$

which gives the claimed result (35). □

Theorem 15. *The following implicit summation formula involving higher-order Lagrange–Hermite–Bernoulli polynomials* ${}_HB_{M,N,n}^{(\alpha|\alpha_1,\alpha_2)}(x|x_1,x_2)$ *and generalized Laguerre-Bernoulli polynomials* ${}_BL_{M,N,n}^{(m)}(x)$ *holds:*

$$\sum_{r=0}^{n} {}_BL_{M,N,n-r}^{(\alpha|m)}(x)\,{}_BL_{M,N,r}^{(\beta|k)}(y)\frac{1}{(n-r)!r!} = {}_HB_{M,N,n}^{(\alpha+\beta|-m-k,0)}(x+y|-1,x_2). \tag{36}$$

Proof. By (13), we write

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HB_{M,N,n}^{(\alpha+\beta|-m-k,0)}(x+y|-1,x_2)t^n &= \frac{1}{{}_1F_1(M;M+N;t)^{\alpha+\beta}}e^{(x+y)t}(1+t)^{m+k} \\ &= \frac{1}{{}_1F_1(M;M+N;t)^{\alpha}}e^{xt}(1+t)^m\frac{1}{{}_1F_1(M;M+N;t)^{\beta}}e^{yt}(1+t)^k \\ &= \left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}(1+t)^m\left(\frac{t}{e^t-1}\right)^{\beta}e^{yt}(1+t)^k \\ &= \sum_{n=0}^{\infty} {}_BL_{M,N,n}^{(\alpha|m)}(x)\frac{t^n}{n!}\sum_{r=0}^{\infty} {}_BL_{M,N,r}^{(\beta|k)}(y)\frac{t^r}{r!} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_LB_{M,N,n-r}^{(\alpha|m)}(x)\,{}_LB_{M,N,r}^{(\beta|k)}(y)\frac{t^n}{(n-r)!r!},\end{aligned}$$

which yields the asserted result (36). □

4. Conclusions

In this paper, we define the multivariable unified Lagrange–Hermite-based hypergeometric Bernoulli polynomials and investigate some of their properties. Then, we derive multifarious connected formulas involving the Miller–Lee polynomials, the Laguerre polynomials, and the Lagrange Hermite–Miller–Lee polynomials. It is demonstrated that the proposed the method allows the derivation of sum rules involving products of generalized polynomials and addition theorems. We developed a point of view based on generating relations, exploited in the past, to study some aspects of the theory of special functions. The possibility of extending the results to include generating functions involving products of Lagrange–Hermite-based hypergeometric Bernoulli polynomials and other polynomials is finally analyzed.

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