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On the Implicit Exponential Finite Difference Method for the Generalized Burgers-Fisher Equation

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Abstract: In this paper, we construct implicit exponential finite difference method to solve the Fisher's, the Burgers-Fisher and the generalized Burgers-Fisher equations with specified initial and boundary conditions. We obtain plots of absolute errors vs x at some values of time t. We compare some of our numerical results with those obtained by other authors using methods such as Adomian Decomposition method, Exp-function method hybridized with Heuristic Computation and Optimal Homotopy Asymptotic method by computing absolute errors at some values of space x and time t.

Keywords: Fisher equation, Burgers-Fisher equation, generalized Burgers-Fisher equation, implicit exponential finite difference method.

1. Introduction

When the problems in various fields as physics, chemistry, biology, mathematics and engineering are modeled, the nonlinear partial differential equations are obtained. The generalized Burgers-Fisher equation is an important nonlinear partial differential equation. Consider a nonlinear partial differential equation of the form,

$$\frac{\partial u}{\partial t}+\alpha u^{\delta}\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=F(u) \tag{1}$$

where $a \leq x \leq b,\ t \geq 0$ with initial condition

$$u\left(x,0\right)=f\left(x\right)$$

and the boundary condition

$$u\left(a,t\right)=g_1\left(t\right),u\left(b,t\right)=g_2\left(t\right).$$

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When $F(u) = \beta u\left(1-u^{\delta}\right)$, Equation (1) gives the generalized Burgers-Fisher equation. Where α, β and δ are parameters such that $\beta \geq 0$, $\delta > 0$. The generalized Burgers-Fisher equation has significant in fluid dynamics, shock wave formation, turbulence, heat conduction, traffic flow, gas dynamics, sound waves in viscous medium, and some other fields of applied science [27]. For $\delta = 1$, the equation is reduced to the Burgers-Fisher equation. When $\alpha = 0$ and $\delta = 1$, the equation becomes the classical Fisher's equation. The equation is described by Fisher's to introduced the propagation of a viral mutant in an infinitely long habitat in 1937. Fisher's equations defined on bounded interval or the whole line has multiple applications area from genes propagation, to tissue engineering, autocatalytic chemical reactions and neurophysiology [1].

For $\beta = 0$ and $\delta = 1$, Equation (1) is called to the Burgers' equation which describes the far field of wave propagation in nonlinear dissipative systems [2].

In literature, many numerical methods have been suggested for approximating solution of the Fisher's equation [1, 3–17, 31]. A restrictive Padé approximation applied for the solution of the generalized Fisher's and Burgers-Fisher equations by Ismail *et al.* [8]. Ismail *et al.* [26] solved the generalized Burgers-Huxley and Burgers-Fisher equations by using the Adomian decomposition method. Mittal and Tripathi [28] proposed a numerical method based on collocation of modified cubic B-spline functions to obtain approximate solution of the generalized Burgers-Fisher and Burgers-Huxley equations. Zarebnia and Jalili [13] solved the generalized Burgers-Fisher and Burgers-Huxley equations with the Chebyshev spectral collocation method. Numerical solution of the generalized Burgers-Fisher equation based on the hybridization of Exp-function method with nature inspired algorithm was constructed by Malik *et al.* [29]. Nawaz *et al.* [27] used optimal homotopy asymptotic method for finding approximate solutions of the generalized Burgers-Huxley and Burgers-Fisher equations. Agbavon *et al.* [30] used NSFD, FTCS methods to obtain numerical solution of the Fisher's equation with coefficient of diffusion term much smaller than coefficient of reaction term. Singh *et al.* [32] used a fourth-order B-spline collocation method for numerical study of the Burgers-Fisher equation. Hussain and Haq proposed a Crank-Nicolson difference scheme combined with meshfree spectral interpolation technique for numerical solution of a class of the Burgers-Fisher type equation [33].

The implicit exponential finite difference method which use in this paper is based on explicit finite difference method is firstly defined by Bhattacharya for solving the heat equation [18]. Bhattacharya [19] and Handschuh and Keith [20] proposed exponential finite difference method for the numerical solution of the Burgers' equation. Bahadir [21] used the exponential finite difference method to solve the KdV equation. After than, İnan and Bahadır [22, 23] proposed implicit, fully implicit and Crank-Nicolson exponential finite difference methods to the Burgers' equation. Also, İnan and Bahadır [24, 25] applied various exponential finite difference methods to the Burgers' equation linearized by Hopf-Cole transformation.

In this paper, we develop an implicit exponential finite difference scheme for solving the Fisher's equation, the Burgers-Fisher equation and the generalized Burgers-Fisher equation. Also, convergence rate is investigated for the generalized Burgers-Fisher equation. Accuracy of the present method is investigated by comparing numerical results obtained by solving test problems with exact solutions.

2. Implicit Exponential Finite Difference Method

In this section, we obtain derivation of the implicit exponential finite difference method when discretized by the generalized Burgers-Fisher equation. The solution domain is discretized into cells described by the nodes set (x_i, t_n) in which $x_i = a + ih$, $(i = 0, 1, 2, ..., N)$ and $t_n = nk$, $(n = 0, 1, 2, ...)$, $h = \Delta x = \frac{b-a}{N}$ is the spatial mesh size and $k = \Delta t$ is the time step. $u(x,t)$ and U_i^n denotes the exact solution and the finite difference approximation to the exact solution, respectively.

We rearrange Equation (1) to obtain

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha u^\delta \frac{\partial u}{\partial x} + F(u). \tag{2}$$

Dividing by u, gives

$$\frac{\partial \ln u}{\partial t} = \frac{1}{u}\left(\frac{\partial^2 u}{\partial x^2} - \alpha u^\delta \frac{\partial u}{\partial x} + F(u)\right). \tag{3}$$

Implicit exponential finite difference method for the generalized Burgers-Fisher equation using the finite difference approximations for derivatives and for $F(u) = \beta u\left(1 - u^\delta\right)$ is given by

$$\begin{aligned} U_i^{n+1} &= U_i^n \exp\left\{\frac{k}{U_i^n}\left[\left(\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2}\right)\right.\right. \\ &\left.\left. -\alpha\left(U_i^{n+1}\right)^\delta\left(\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2h}\right) + \beta U_i^{n+1}\left(1 - \left(U_i^{n+1}\right)^\delta\right)\right]\right\}. \end{aligned} \tag{4}$$

For $\alpha = 0$ and $\delta = \beta = 1$, Equation (4) describes the implicit exponential finite difference method for the Fishers' equation. For $\delta = 1$, Equation (4) gives implicit exponential finite difference method for the Burgers-Fisher equation.

In this paper, to measure the accuracy of the method is used the absolute error which defined by

$$\left|u\left(x_i, t_n\right) - U\left(x_i, t_n\right)\right|.$$

When the implicit exponential finite difference method is applied to the equations, a system of nonlinear difference equations is obtained. These nonlinear difference equations systems are solved by Newton' s method.

3. Numerical Results for the Fisher's Equation

Example 3.1

Consider the Fisher's equation of the form;

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=\beta u\left(1-u\right),\quad 0\le x\le 1,\ t\ge 0 \tag{5}$$

with initial conditions

$$u\left(x,0\right)=\frac{1}{4}\left[1-\tanh\left(\frac{x}{2\sqrt{6}}\right)\right]^2$$

and boundary conditions

$$u\left(0,t\right)=\frac{1}{4}\left[1-\tanh\left(-\frac{5}{12}t\right)\right]^2$$

and

$$u\left(1,t\right)=\frac{1}{4}\left[1-\tanh\left(\frac{1}{2\sqrt{6}}-\frac{5}{12}t\right)\right]^2.$$

The exact solution, with $\beta=1$ is

$$u\left(x,t\right)=\frac{1}{4}\left[1-\tanh\left(\frac{1}{2\sqrt{6}}\left(x-\frac{5}{\sqrt{6}}t\right)\right)\right]^2. \tag{6}$$

Table 1 shows the numerical solutions, exact solutions and absolute errors at some values of x and t using $k=0.00001$ and $h=0.01$. In Table 2, we present maximum absolute errors obtained by the method for different values of h and k. Figure 1 shows exact and numerical solutions vs $x\in[0,1]$ vs $t\in[0,10]$ using $k=0.0001$ and $h=0.01$. To see behaviors, absolute errors vs x are displayed for different times in Figure 2 using $k=0.0001$ and $h=0.01$.

Table 1: Numerical solutions obtained using implicit exponential finite difference scheme for Example 3.1

t	x	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.24194381	0.24194382	9.09795x10^{-9}
	0.5	0.20358872	0.20358873	1.29159x10^{-8}
	0.9	0.16906886	0.16906887	9.26263x10^{-9}
1	0.1	0.47385284	0.47385288	3.59326x10^{-8}
	0.5	0.42550850	0.42550861	1.08649x10^{-7}
	0.9	0.37750606	0.37750611	4.10260x10^{-8}
10	0.1	0.99949942	0.99949942	9.17674x10^{-11}
	0.5	0.99941066	0.99941066	2.68485x10^{-10}
	0.9	0.99930617	0.99930617	1.02407x10^{-10}

Table 2: Maximum absolute errors obtained using implicit exponential finite difference scheme for Example 3.1

	$k = 0.001$		$k = 0.0001$		$k = 0.000001$	
t	$h = 0.05$	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	$h = 0.1$
0.0001	–	–	$1.19454\text{x}10^{-9}$	$7.18609\text{x}10^{-10}$	$1.47430\text{x}10^{-10}$	$6.29750\text{x}10^{-10}$
0.0005	–	–	$5.97340\text{x}10^{-9}$	$3.59162\text{x}10^{-9}$	$7.37136\text{x}10^{-10}$	$3.14866\text{x}10^{-9}$
0.001	$1.34075\text{x}10^{-7}$	$1.29241\text{x}10^{-7}$	$1.19483\text{x}10^{-8}$	$7.17503\text{x}10^{-9}$	$1.47424\text{x}10^{-9}$	$6.29725\text{x}10^{-9}$
0.01	$1.34134\text{x}10^{-6}$	$1.29132\text{x}10^{-6}$	$1.19598\text{x}10^{-7}$	$7.12699\text{x}10^{-8}$	$1.47298\text{x}10^{-8}$	$6.28813\text{x}10^{-8}$
0.1	$1.04182\text{x}10^{-5}$	$1.00147\text{x}10^{-5}$	$9.33123\text{x}10^{-7}$	$5.59419\text{x}10^{-7}$	$1.13330\text{x}10^{-7}$	$4.83340\text{x}10^{-7}$
0.2	$1.45804\text{x}10^{-5}$	$1.40511\text{x}10^{-5}$	$1.30752\text{x}10^{-6}$	$7.95379\text{x}10^{-7}$	$1.55452\text{x}10^{-7}$	$6.65058\text{x}10^{-7}$
0.5	$1.57046\text{x}10^{-5}$	$1.52557\text{x}10^{-5}$	$1.43322\text{x}10^{-6}$	$9.72046\text{x}10^{-7}$	$1.38450\text{x}10^{-7}$	$5.99612\text{x}10^{-7}$
1.0	$1.08894\text{x}10^{-5}$	$1.08807\text{x}10^{-5}$	$1.08209\text{x}10^{-6}$	$1.05855\text{x}10^{-6}$	$1.21106\text{x}10^{-8}$	$4.77820\text{x}10^{-8}$
2.0	$4.26562\text{x}10^{-7}$	$8.09925\text{x}10^{-7}$	$1.62078\text{x}10^{-7}$	$5.81767\text{x}10^{-7}$	$1.41532\text{x}10^{-7}$	$5.65655\text{x}10^{-7}$
5.0	$1.39935\text{x}10^{-6}$	$1.44516\text{x}10^{-6}$	$1.53753\text{x}10^{-7}$	$1.99853\text{x}10^{-7}$	$1.67674\text{x}10^{-8}$	$6.28945\text{x}10^{-8}$

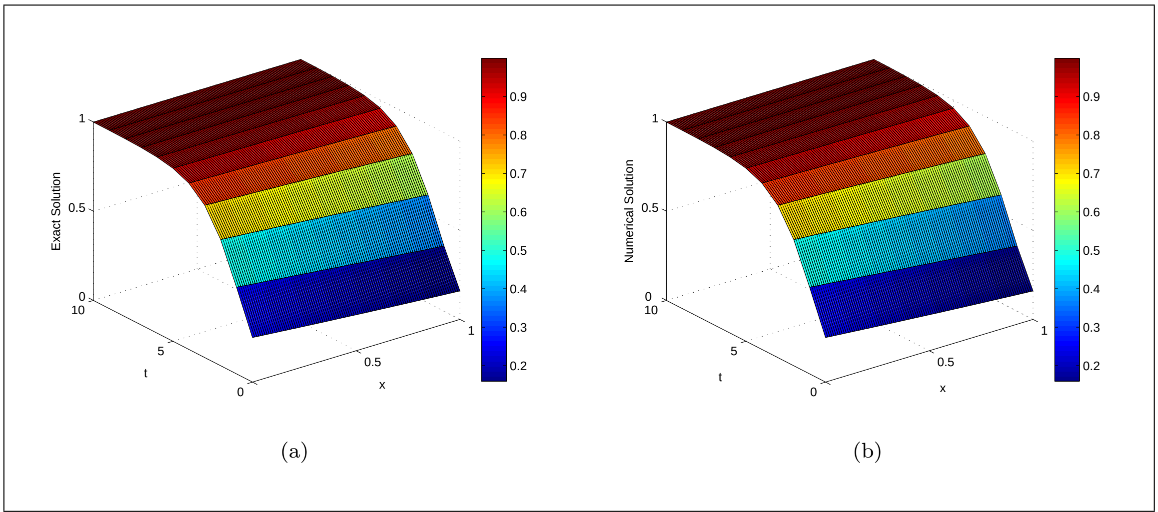


Figure 1: (a) Exact and (b) Numerical solutions vs $x \in [0, 1]$ vs $t \in [0, 10]$

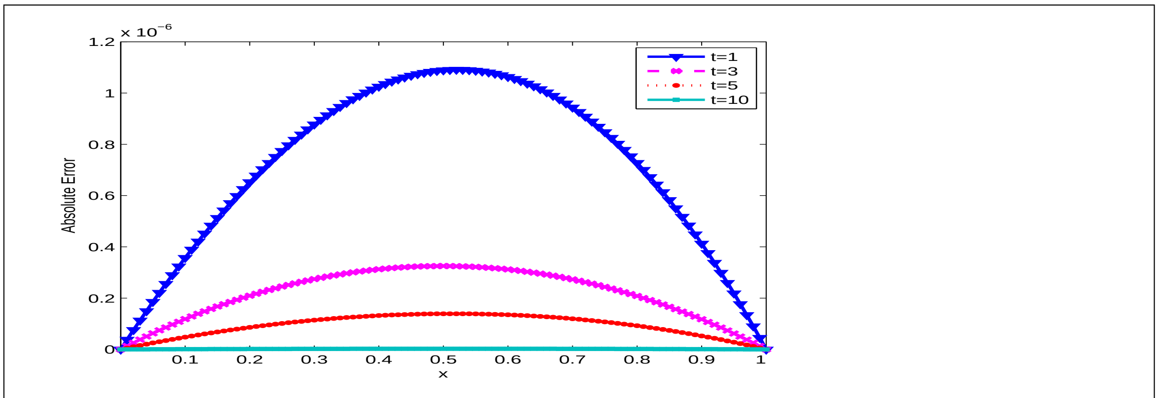


Figure 2: Absolute errors at various time levels using present method for Example 3.1

Example 3.2

Consider the Fisher's equation of the form;

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \beta u \left(1 - u\right), \quad -1 \leq x \leq 1, \ t \geq 0 \tag{7}$$

with the exact solution of this equation is given by

$$u(x,t) = \frac{1}{\left[1 + \exp\left(\sqrt{\frac{\beta}{6}}x - \left(\frac{5\beta}{6}\right)t\right)\right]^2}. \quad (8)$$

Table 3-5 show the numerical solutions, exact solutions and absolute errors for various values of x and t and different values of β for $k = 0.0001$ and $h = 0.1$. Figures 3-5 show exact and numerical solutions vs $x \in [0,1]$ vs $t \in [0,10]$ for different values of $\beta = 0.2$, $\beta = 1$, $\beta = 5$ and using $k = 0.0001$ and $h = 0.02$, respectively. To see behaviors, absolute errors are exhibited for different times in Figures 6-8 for different values of $\beta = 0.2$, $\beta = 1$, $\beta = 5$ and $k = 0.0001$ and $h = 0.02$, respectively.

Table 3: Numerical solutions obtained using implicit exponential finite difference scheme for Example 3.2 with $\beta = 0.2$

	$t = 0.001$			$t = 10$		
x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
−0.9	0.2927117284	0.2927117287	2.86702x10^{-10}	0.7428351350	0.7428351438	8.78714x10^{-9}
−0.7	0.2829692727	0.2829692730	2.98866x10^{-10}	0.7352621172	0.7352621421	2.48827x10^{-8}
−0.5	0.2733695269	0.2733695272	2.95791x10^{-10}	0.7275291576	0.7275291960	3.84063x10^{-8}
−0.3	0.2639198250	0.2639198253	2.92981x10^{-10}	0.7196367882	0.7196368369	4.86864x10^{-8}
−0.1	0.2546271053	0.2546271055	2.90501x10^{-10}	0.7115858154	0.7115858705	5.51091x10^{-8}
0.0	0.2500416684	0.2500416687	2.89397x10^{-10}	0.7075011827	0.7075012394	5.66983x10^{-8}
0.1	0.2454978890	0.2454978892	2.88389x10^{-10}	0.7033773321	0.7033773892	5.71202x10^{-8}
0.3	0.2365382619	0.2365382622	2.86677x10^{-10}	0.6950127290	0.6950127833	5.42287x10^{-8}
0.5	0.2277538582	0.2277538584	2.85387x10^{-10}	0.6864937060	0.6864937520	4.60093x10^{-8}
0.7	0.2191498464	0.2191498466	2.84516x10^{-10}	0.6778222819	0.6778223141	3.21064x10^{-8}
0.9	0.2107309187	0.2107309190	2.69627x10^{-10}	0.6690008048	0.6690008170	1.22381x10^{-8}

Table 4: Numerical solutions obtained using implicit exponential finite difference scheme for Example 3.2 with $\beta = 1$

	$t = 0.001$			$t = 10$		
x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
−0.9	0.3493255528	0.3493255604	7.63782x10^{-9}	0.9996671635	0.9996671613	2.22869x10^{-9}
−0.7	0.3262298018	0.3262298097	7.85682x10^{-9}	0.9996388549	0.9996388488	6.11627x10^{-9}
−0.5	0.3036679500	0.3036679577	7.65473x10^{-9}	0.9996081393	0.9996081301	9.21058x10^{-9}
−0.3	0.2817358527	0.2817358602	7.46479x10^{-9}	0.9995748122	0.9995748007	1.14618x10^{-8}
−0.1	0.2605214660	0.2605214733	7.30433x10^{-9}	0.9995386516	0.9995386388	1.28085x10^{-8}
0.0	0.2502083767	0.2502083840	7.23947x10^{-9}	0.9995194343	0.9995194212	1.31200x10^{-8}
0.1	0.2401033892	0.2401033964	7.18659x10^{-9}	0.9994994169	0.9994994037	1.31765x10^{-8}
0.3	0.2205496633	0.2205496704	7.12017x10^{-9}	0.9994568468	0.9994568343	1.24779x10^{-8}
0.5	0.2019168663	0.2019168734	7.10860x10^{-9}	0.9994106581	0.9994106475	1.06099x10^{-8}
0.7	0.1842495294	0.1842495366	7.14981x10^{-9}	0.9993605436	0.9993605361	7.45377x10^{-9}
0.9	0.1675798859	0.1675798927	6.86638x10^{-9}	0.9993061698	0.9993061669	2.87276x10^{-9}

Table 5: Numerical solutions obtained using implicit exponential finite difference scheme for Example 3.2 with $\beta = 5$

	$t = 0.001$			$t = 10$		
x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
-0.9	0.4836585569	0.4836587509	1.93962x10^{-7}	1.0000000000	0.9999999999	2.09832x10^{-14}
-0.7	0.4296423866	0.4296425968	2.10140x10^{-7}	1.0000000000	0.9999999999	5.10703x10^{-14}
-0.5	0.3759615173	0.3759617234	2.06147x10^{-7}	1.0000000000	0.9999999999	6.93889x10^{-14}
-0.3	0.3238323364	0.3238325326	1.96142x10^{-7}	1.0000000000	0.9999999999	8.00471x10^{-14}
-0.1	0.2744139373	0.2744141229	1.85532x10^{-7}	1.0000000000	0.9999999999	8.48210x10^{-14}
0.0	0.2510427502	0.2510429318	1.81553x10^{-7}	1.0000000000	0.9999999999	8.53762x10^{-14}
0.1	0.2287074324	0.2287076115	1.79037x10^{-7}	1.0000000000	0.9999999999	8.49321x10^{-14}
0.3	0.1874734720	0.1874736510	1.79020x10^{-7}	1.0000000000	0.9999999999	8.03802x10^{-14}
0.5	0.1511821391	0.1511823239	1.84810x10^{-7}	1.0000000000	0.9999999999	6.97220x10^{-14}
0.7	0.1200021179	0.1200023112	1.93262x10^{-7}	1.0000000000	0.9999999999	5.10703x10^{-14}
0.9	0.0938271626	0.0938273526	1.89955x10^{-7}	1.0000000000	0.9999999999	2.10942x10^{-14}

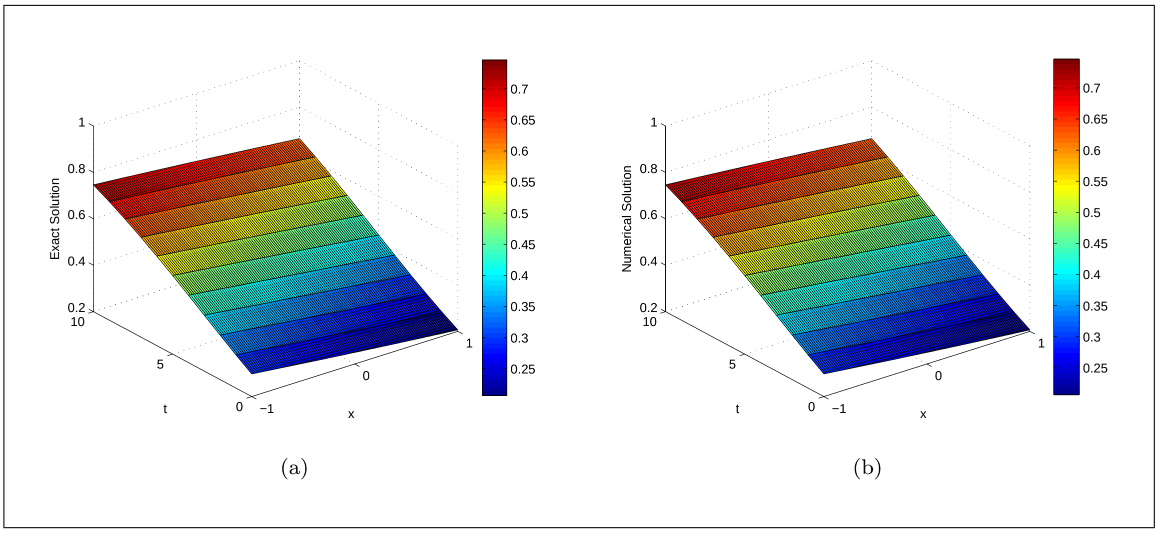


Figure 3: (a) Exact and (b) Numerical solutions vs $x \in [-1, 1]$ vs $t \in [0, 10]$

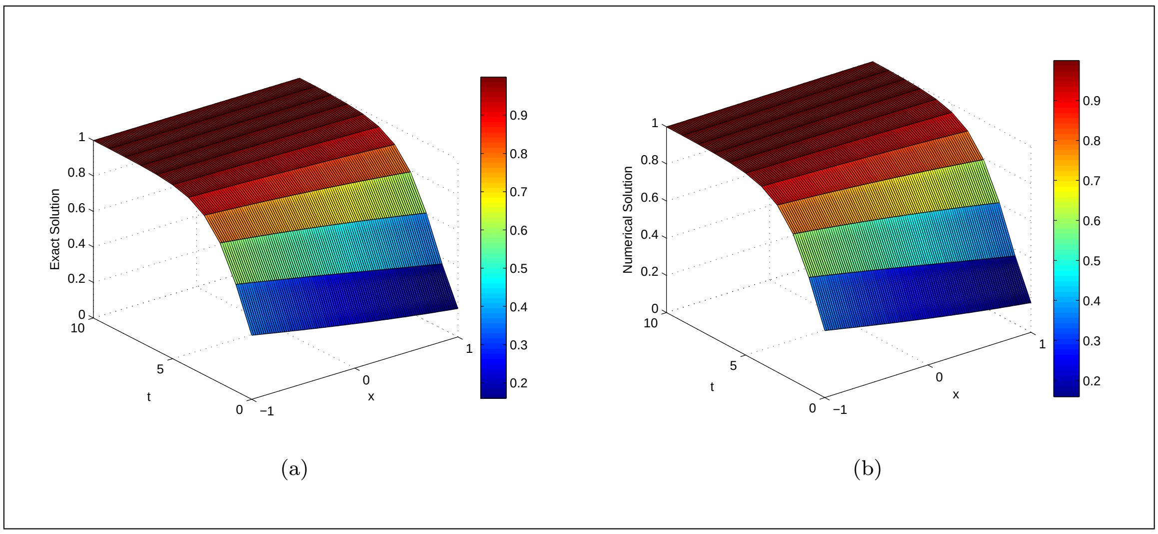


Figure 4: (a) Exact and (b) Numerical solutions vs $x \in [-1, 1]$ vs $t \in [0, 10]$

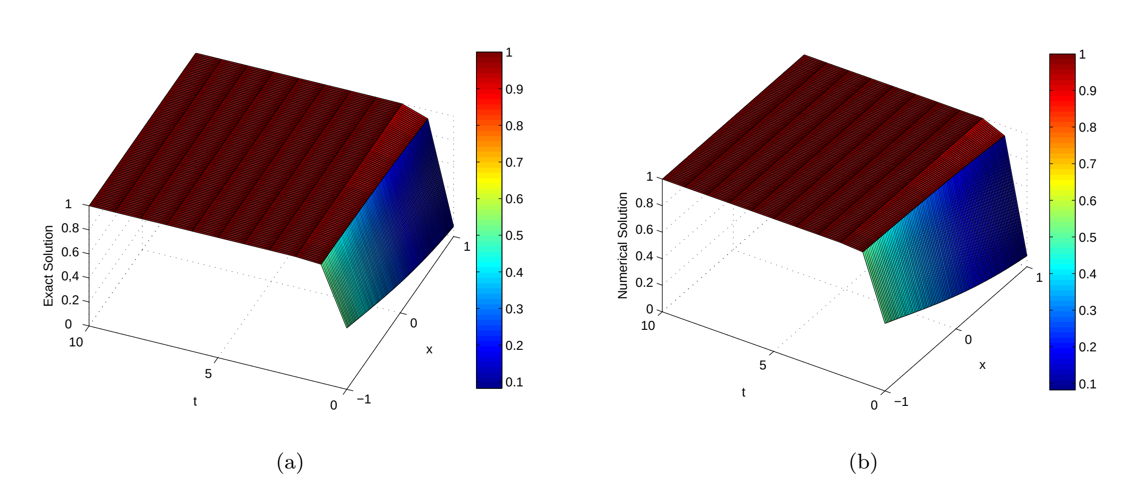


Figure 5: (a) Exact and (b) Numerical solutions vs $x \in [-1, 1]$ vs $t \in [0, 10]$

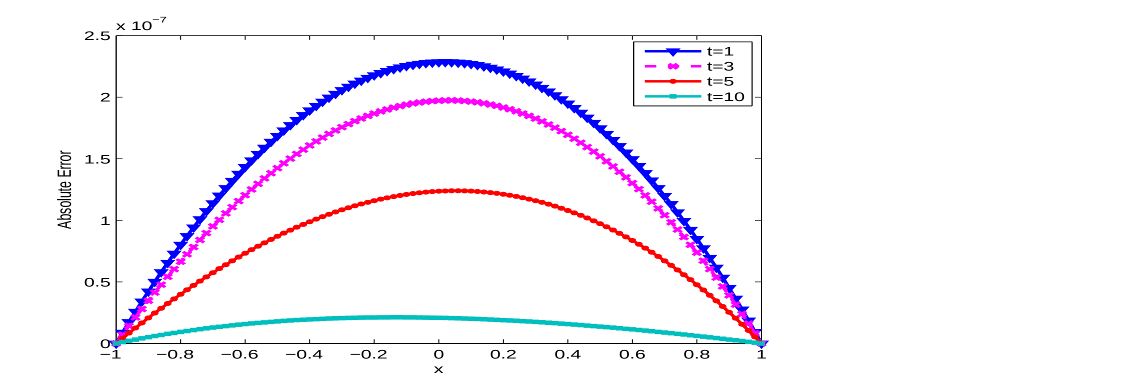


Figure 6: Absolute errors at various time levels using present method for $\beta = 0.2$

4. Numerical Results for the Burgers-Fisher Equation

The Burgers-Fisher equation is given by

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u\,(1-u)\,, 0 \leq x \leq 1, t \geq 0. \tag{9}$$

with initial conditions

$$u\,(x, 0) = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{-\alpha x}{4}\right]$$

and boundary conditions

$$u\,(0, t) = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{(\alpha^2 + 2\beta)}{8} t\right]$$

and

$$u\,(1, t) = \frac{1}{2} + \frac{1}{2}\tanh\left[(\frac{\alpha^2 + 2\beta}{8})t - \frac{\alpha}{4}\right].$$

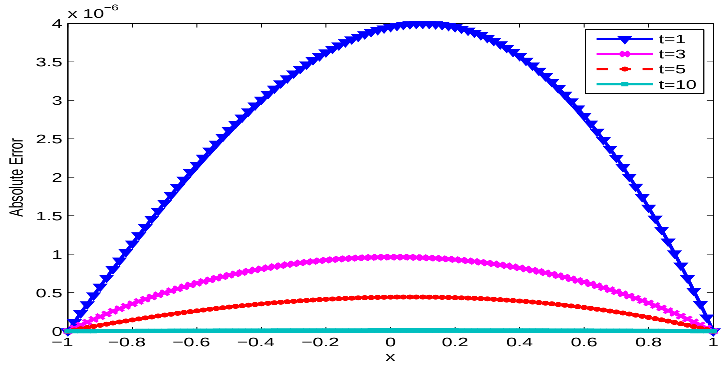


Figure 7: Absolute errors at various time levels using present method for $\beta = 1$

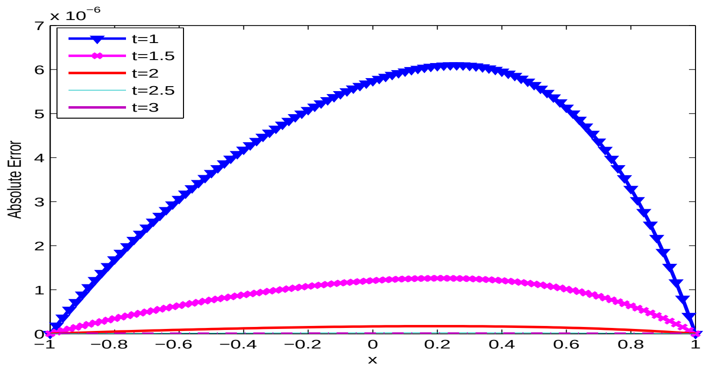


Figure 8: Absolute errors at various time levels using present method for $\beta = 5$

The exact solution of this equation is given by

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{-\alpha}{4}\left(x - (\frac{\alpha^2 + 2\beta}{2\alpha})t\right)\right]. \tag{10}$$

We solve the Burgers-Fisher equation for following different values of α and β. Table 6-8 show the numerical solutions, exact solutions and absolute errors for various values of x, t with $\alpha = \beta = 0.001$ and $\alpha = \beta = 0.5$. Table 9-11 show the numerical solutions, exact solutions and absolute errors for various values of x, t with $\alpha > \beta$ $(\alpha = 1,\ \beta = 0.5)$ and $\alpha < \beta$ $(\alpha = 1,\ \beta = 5)$. In Table 12, we compare absolute errors obtained by the present method with the other methods [26, 27, 29] for $h = 0.1$ and $k = 0.001$. To see behaviors, absolute errors at various time levels using present method for $\alpha = \beta$, $\alpha < \beta$ and $\alpha > \beta$ are exhibited for different times at from $t = 0$ to $t = 5$ in Figures 9-10 for $h = 0.01$ and $k = 0.0001$.

Table 6: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.001$

		$\alpha = \beta = 0.001$			$\alpha = \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.499990	0.499990	6.75349x10^{-13}	0.495156	0.495157	2.02488x10^{-7}
	0.5	0.499940	0.499940	1.74344x10^{-12}	0.470192	0.470192	5.18989x10^{-7}
	0.9	0.499890	0.499890	6.75626x10^{-13}	0.445375	0.445375	2.34788x10^{-7}
1	0.1	0.500238	0.500238	7.17337x10^{-12}	0.631231	0.631232	1.61229x10^{-6}
	0.5	0.500188	0.500188	2.21669x10^{-11}	0.607663	0.607669	5.54168x10^{-6}
	0.9	0.500138	0.500138	7.17515x10^{-12}	0.583583	0.583585	1.84876x10^{-6}
10	0.1	0.502488	0.502488	7.15528x10^{-12}	0.996316	0.996316	1.16984x10^{-8}
	0.5	0.502438	0.502438	2.21255x10^{-11}	0.995930	0.995930	5.61914x10^{-8}
	0.9	0.502388	0.502388	7.15650x10^{-12}	0.995504	0.995504	1.42948x10^{-8}

Table 7: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.01$

		$\alpha = \beta = 0.001$			$\alpha = \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.499990000625	0.499990000629	3.86408x10^{-12}	0.495156	0.495158	1.27465x10^{-6}
	0.5	0.499940000625	0.499940000631	6.15369x10^{-12}	0.470192	0.470194	2.28048x10^{-6}
	0.9	0.499890000627	0.499890000631	3.86619x10^{-12}	0.445375	0.445377	1.58862x10^{-6}
1	0.1	0.500237562482	0.500237562550	6.78532x10^{-11}	0.631231	0.631246	1.52795x10^{-5}
	0.5	0.500187562491	0.500187562704	2.13117x10^{-10}	0.607663	0.607717	5.35838x10^{-5}
	0.9	0.500137562497	0.500137562564	6.78673x10^{-11}	0.583583	0.583601	1.75688x10^{-5}
10	0.1	0.502488104462	0.502488104530	6.76663x10^{-11}	0.996316	0.996316	8.10339x10^{-8}
	0.5	0.502438105676	0.502438105888	2.12696x10^{-10}	0.995930	0.995930	4.58615x10^{-7}
	0.9	0.502388106840	0.502388106908	6.76804x10^{-11}	0.995504	0.995504	9.90982x10^{-8}

Table 8: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.1$

		$\alpha = \beta = 0.001$			$\alpha = \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
1	0.1	0.500237562482	0.500237562902	4.20209x10^{-10}	0.631231	0.631293	6.21407x10^{-5}
	0.5	0.500187562491	0.500187563859	1.367691x10^{-9}	0.607663	0.607968	3.04568x10^{-4}
	0.9	0.500137562497	0.500137562911	4.20311x10^{-10}	0.583583	0.583653	7.03671x10^{-5}
10	0.1	0.502488104462	0.502488104901	4.18467x10^{-10}	0.996316	0.996315	6.44044x10^{-7}
	0.5	0.502438105676	0.502438107039	1.36357x10^{-9}	0.995930	0.995929	5.48251x10^{-7}
	0.9	0.502388106840	0.502388107259	4.18569x10^{-10}	0.995504	0.995503	7.83912x10^{-7}

Table 9: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.001$

		$\alpha = 1,\ \beta = 5$			$\alpha = 1, \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.500625	0.500643	1.84502×10^{-5}	0.489377	0.489377	2.56357×10^{-7}
	0.5	0.450785	0.450834	4.87630×10^{-5}	0.439670	0.439671	6.98109×10^{-7}
	0.9	0.401913	0.401937	2.36744×10^{-5}	0.391145	0.391146	3.60066×10^{-7}
1	0.1	0.994514	0.994515	1.74290×10^{-6}	0.668188	0.668189	1.76625×10^{-6}
	0.5	0.993307	0.993316	9.04853×10^{-6}	0.622459	0.622466	6.68757×10^{-6}
	0.9	0.991837	0.991840	2.60564×10^{-6}	0.574442	0.574445	2.35013×10^{-6}
10	0.1	0.994514	0.994515	1.74290×10^{-6}	0.999419	0.999419	4.34509×10^{-9}
	0.5	0.993307	0.993316	9.04853×10^{-6}	0.999290	0.999290	1.94902×10^{-8}
	0.9	0.991837	0.991840	2.60564×10^{-6}	0.999133	0.999133	6.48182×10^{-9}

Table 10: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.01$

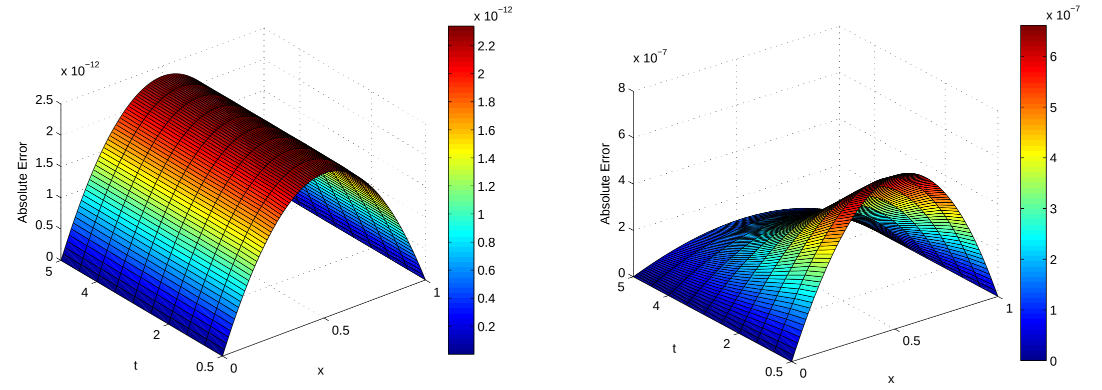
		$\alpha = 1,\ \beta = 5$			$\alpha = 1, \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.500625	0.500738	1.12634×10^{-4}	0.489377	0.489379	2.29972×10^{-6}
	0.5	0.450785	0.451006	2.21064×10^{-4}	0.439670	0.439675	4.51812×10^{-6}
	0.9	0.401913	0.402079	1.65910×10^{-4}	0.391145	0.391149	3.32034×10^{-6}
1	0.1	0.994514	0.994530	1.62856×10^{-5}	0.668188	0.668206	1.88226×10^{-5}
	0.5	0.993307	0.993397	8.95713×10^{-5}	0.622459	0.622532	7.25601×10^{-5}
	0.9	0.991837	0.991862	2.43565×10^{-5}	0.574442	0.574468	2.56617×10^{-5}
10	0.1	1.000000	0.999999	2.2204×10^{-16}	0.999419	0.999419	2.19012×10^{-9}
	0.5	1.000000	0.999999	4.4409×10^{-16}	0.999290	0.999290	5.66220×10^{-8}
	0.9	1.000000	0.999999	2.2204×10^{-16}	0.999133	0.999133	3.29774×10^{-9}

Table 11: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.1$

		$\alpha = 1,\ \beta = 5$			$\alpha = 1, \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
1	0.1	0.994514	0.994659	1.45399×10^{-4}	0.668188	0.668240	5.17464×10^{-5}
	0.5	0.993307	0.994229	9.22135×10^{-4}	0.622459	0.622847	3.87928×10^{-4}
	0.9	0.991837	0.992054	2.16160×10^{-4}	0.574442	0.574504	6.16300×10^{-5}
10	0.1	1.000000	1.000000	0.00000	0.999419	0.999418	3.19818×10^{-7}
	0.5	1.000000	1.000000	0.00000	0.999290	0.999289	7.44709×10^{-7}
	0.9	1.000000	1.000000	0.00000	0.999133	0.999133	4.76764×10^{-7}

Table 12: Absolute error obtained using implicit exponential finite difference scheme with $\alpha = \beta = 0.001$

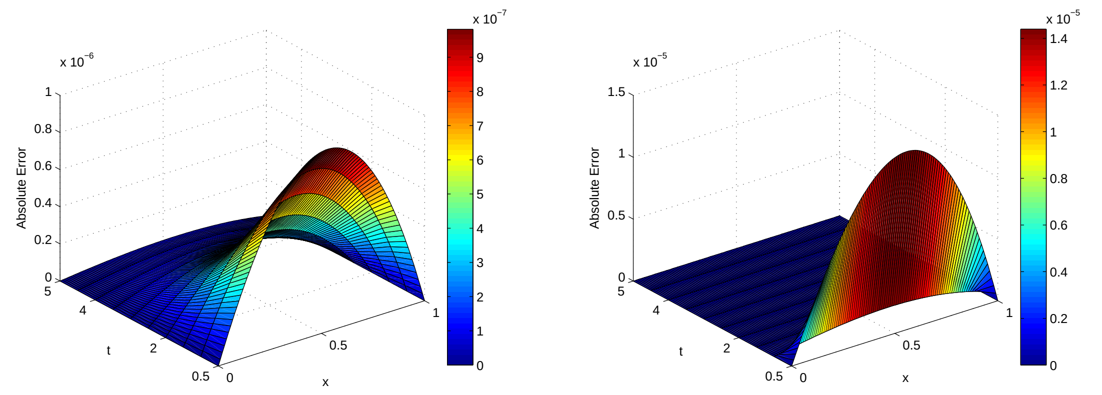
x	t	Absolute Error	[26]	[27]	[29]
0.1	0.001	5.72875×10^{-14}	1.93753×10^{-6}	2.24513×10^{-8}	1.97×10^{-8}
	0.005	3.22686×10^{-13}	9.68763×10^{-6}	1.12257×10^{-7}	1.97×10^{-8}
	0.01	6.75349×10^{-13}	1.93752×10^{-5}	2.24514×10^{-7}	1.97×10^{-8}
0.5	0.001	6.25056×10^{-14}	1.93738×10^{-6}	4.57775×10^{-8}	3.58×10^{-9}
	0.005	8.12184×10^{-13}	9.68691×10^{-6}	2.28888×10^{-7}	3.71×10^{-9}
	0.01	1.74344×10^{-12}	1.93738×10^{-5}	4.57777×10^{-7}	3.88×10^{-9}
0.9	0.001	5.72875×10^{-14}	1.93724×10^{-6}	4.57775×10^{-7}	1.80×10^{-8}
	0.005	3.22742×10^{-13}	9.68619×10^{-6}	2.28888×10^{-7}	1.77×10^{-8}
	0.01	6.75626×10^{-13}	1.93724×10^{-5}	4.57777×10^{-7}	1.74×10^{-8}



(a) $\alpha = \beta = 0.001$.

(b) $\alpha = \beta = 0.5$.

Figure 9: Absolute errors at various time levels using present method for $\alpha = \beta = 0.001$ and $\alpha = \beta = 0.5$



(a) $\alpha = 1.0, \beta = 0.5$.

(b) $\alpha = 1.0, \beta = 5.0$.

Figure 10: Absolute errors at various time levels using present method for $\alpha = 1,\ \beta = 0.5$ and $\alpha = 1,\ \beta = 5$

5. Numerical Results for the Generalized Burgers-Fisher Equation

The generalized Burgers-Fisher equation is

$$\frac{\partial u}{\partial t} + \alpha u^{\delta} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u \left(1 - u^{\delta}\right), 0 \le x \le 1, t \ge 0 \tag{11}$$

and the exact solution for the equation is given by

$$u\left(x,t\right) = \left(\frac{1}{2} + \frac{1}{2} \tanh\left[\Psi_1 \left(x - \ \Psi_2 t\right)\right]\right)^{\frac{1}{\delta}} \tag{12}$$

where

$$\Psi_1 = \frac{-\alpha\delta}{2(\delta+1)} \text{ and } \Psi_2 = \frac{\alpha}{\delta+1} + \frac{\beta(\delta+1)}{\alpha}.$$

The accuracy of the method is measured in terms of the error norm defined by

$$E = \left[\frac{\sum_{i=0}^{N} |u_i - U_i|^2}{\sum_{i=0}^{N} |u_i|^2} \right]^{\frac{1}{2}}. \tag{13}$$

The rates of convergence of the method, computed using

$$Convergence\ Rate = \frac{\log\left(E^h/E^{h/2}\right)}{\log(2)} \tag{14}$$

where E^h and $E^{h/2}$ are the errors defined in Equation (13) with the grid size h and $h/2$, respectively.

We solved the generalized Burgers-Fisher equation for following different values of α and β and $\delta = 2$. Table 13-15 show the numerical solutions, exact solutions and absolute errors for various values of x, t with $\alpha = \beta$ for $\delta = 2$, $h = 0.1$ and $k = 0.001$, $k = 0.01$, $k = 0.1$, respectively. Table 16-18 show the numerical solutions, exact solutions and absolute errors for various values of x, t with $\alpha < \beta$ and $\alpha > \beta$ for $\delta = 2$, $h = 0.1$ and $k = 0.001$, $k = 0.01$, $k = 0.1$, respectively. In Table 19, we compare absolute errors obtained by the present method with the other methods [26, 27, 29] for $h = 0.1$ and $k = 0.00001$. To see behaviors, absolute errors at various time levels using present method for $\alpha = \beta$, $\alpha < \beta$ and $\alpha > \beta$ are exhibited for different times at from $t = 0$ to $t = 5$ in Figures 11-12 for $h = 0.01$ and $k = 0.0001$.

Table 13: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.001$

		$\delta = 2, \alpha = \beta = 0.1$			$\delta = 2, \alpha = \beta = 1$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.706285	0.706285	7.03520x10^{-9}	0.699207	0.699208	7.93053x10^{-7}
	0.5	0.701550	0.701550	3.23084x10^{-8}	0.650264	0.650268	3.77421x10^{-6}
	0.9	0.696785	0.696785	7.60688x10^{-9}	0.599537	0.599538	1.44137x10^{-6}
1	0.1	0.740766	0.740766	1.14565x10^{-7}	0.946671	0.946675	3.18359x10^{-6}
	0.5	0.736289	0.736290	3.88604x10^{-8}	0.932003	0.932017	1.49506x10^{-5}
	0.9	0.731776	0.731776	1.17706x10^{-7}	0.913839	0.913843	4.66859x10^{-6}
10	0.1	0.939372	0.939372	3.01990x10^{-8}	0.999999	0.999999	7.77156x10^{-15}
	0.5	0.937884	0.937884	1.21356x10^{-7}	0.999999	0.999999	3.79696x10^{-14}
	0.9	0.936361	0.936362	3.13935x10^{-8}	0.999999	0.999999	1.14353x10^{-14}

Table 14: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.01$

		$\delta = 2, \alpha = \beta = 0.1$			$\delta = 2, \alpha = \beta = 1$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.7062852759	0.7062852769	$9.95526 \text{x} 10^{-10}$	0.699207	0.699208	$1.15896 \text{x} 10^{-6}$
	0.5	0.7015503273	0.7015503327	$5.42282 \text{x} 10^{-9}$	0.650264	0.650271	$6.82174 \text{x} 10^{-6}$
	0.9	0.6967851996	0.6967852057	$6.12116 \text{x} 10^{-9}$	0.599537	0.599544	$7.74007 \text{x} 10^{-6}$
1	0.1	0.7407656051	0.7407666570	$1.05186 \text{x} 10^{-6}$	0.946671	0.946696	$2.47461 \text{x} 10^{-5}$
	0.5	0.7362894454	0.7362931069	$3.66157 \text{x} 10^{-6}$	0.932003	0.932131	$1.28498 \text{x} 10^{-4}$
	0.9	0.7317757472	0.7317768282	$1.08096 \text{x} 10^{-6}$	0.913839	0.913875	$3.65420 \text{x} 10^{-5}$
10	0.1	0.9393724828	0.9393727397	$2.56852 \text{x} 10^{-7}$	0.999999	0.999999	$5.04041 \text{x} 10^{-14}$
	0.5	0.9378835417	0.9378846395	$1.09784 \text{x} 10^{-6}$	0.999999	0.999999	$2.98761 \text{x} 10^{-13}$
	0.9	0.9363617072	0.9363619740	$2.66853 \text{x} 10^{-7}$	0.999999	0.999999	$7.23865 \text{x} 10^{-14}$

Table 15: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.1$

		$\delta = 2, \alpha = \beta = 0.1$			$\delta = 2, \alpha = \beta = 1$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
1	0.1	0.740766	0.740770	$4.25689 \text{x} 10^{-6}$	0.946671	0.946684	$1.29093 \text{x} 10^{-5}$
	0.5	0.736289	0.736307	$1.75738 \text{x} 10^{-5}$	0.932002	0.932428	$4.24973 \text{x} 10^{-4}$
	0.9	0.731776	0.731780	$4.40233 \text{x} 10^{-6}$	0.913839	0.913838	$5.01805 \text{x} 10^{-7}$
10	0.1	0.939372	0.939372	$3.16261 \text{x} 10^{-7}$	0.999999	0.999999	$1.62870 \text{x} 10^{-13}$
	0.5	0.937883	0.937885	$1.59124 \text{x} 10^{-6}$	0.999999	0.999999	$4.26881 \text{x} 10^{-13}$
	0.9	0.936362	0.936361	$3.28356 \text{x} 10^{-7}$	0.999999	0.999999	$3.20854 \text{x} 10^{-13}$

Table 16: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.001$

		$\delta = 2, \alpha = 1, \beta = 5$			$\delta = 2, \alpha = 1, \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.713363829027	0.713381730891	$1.79019 \text{x} 10^{-5}$	0.697418274826	0.697418461423	$1.86597 \text{x} 10^{-6}$
	0.5	0.665190218322	0.665277650772	$8.74324 \text{x} 10^{-5}$	0.648386447615	0.648387426805	$9.79190 \text{x} 10^{-7}$
	0.9	0.614868312087	0.614900877457	$3.25654 \text{x} 10^{-5}$	0.597616011797	0.597616366760	$3.54962 \text{x} 10^{-7}$
1	0.1	0.999980570769	0.999980597216	$2.64470 \text{x} 10^{-8}$	0.872080385273	0.872082457660	$2.07239 \text{x} 10^{-6}$
	0.5	0.999974633322	0.999974769605	$1.36283 \text{x} 10^{-7}$	0.841819948106	0.841828571806	$8.62370 \text{x} 10^{-6}$
	0.9	0.999966881519	0.999966920732	$3.92133 \text{x} 10^{-8}$	0.806674525107	0.806677340997	$2.81589 \text{x} 10^{-6}$
10	0.1	1.000000000000	0.999999999999	$9.99201 \text{x} 10^{-16}$	0.999997370473	0.999997370540	$6.67595 \text{x} 10^{-11}$
	0.5	1.000000000000	0.999999999999	$2.66454 \text{x} 10^{-15}$	0.999996566880	0.999996567183	$3.02891 \text{x} 10^{-10}$
	0.9	1.000000000000	0.999999999999	$9.99201 \text{x} 10^{-16}$	0.999995517708	0.999995517809	$1.00871 \text{x} 10^{-10}$

Table 17: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.01$

		$\delta = 2, \alpha = 1, \beta = 5$			$\delta = 2, \alpha = 1, \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
0.01	0.1	0.713364	0.713368	$4.31389\text{x}10^{-6}$	0.697418274825	0.697418628996	$3.54170\text{x}10^{-7}$
	0.5	0.665190	0.665281	$9.10517\text{x}10^{-5}$	0.648386447615	0.648388523486	$2.07587\text{x}10^{-6}$
	0.9	0.614868	0.615014	$1.45739\text{x}10^{-4}$	0.597616011797	0.597618327134	$2.31534\text{x}10^{-6}$
1	0.1	0.999981	0.999981	$2.85038\text{x}10^{-7}$	0.872080385273	0.872095937360	$1.55521\text{x}10^{-5}$
	0.5	0.999975	0.999976	$1.49891\text{x}10^{-6}$	0.841819948106	0.841892335589	$7.23875\text{x}10^{-5}$
	0.9	0.999967	0.999967	$4.18801\text{x}10^{-7}$	0.806674525107	0.806697007733	$2.24826\text{x}10^{-5}$
10	0.1	1.000000	0.999999	$2.22040\text{x}10^{-16}$	0.999997370473	0.999997370711	$2.37786\text{x}10^{-10}$
	0.5	1.000000	0.999999	$4.44091\text{x}10^{-16}$	0.999996566880	0.999996568444	$1.56450\text{x}10^{-9}$
	0.9	1.000000	0.999999	$2.22041\text{x}10^{-16}$	0.999995517708	0.999995518042	$3.33484\text{x}10^{-10}$

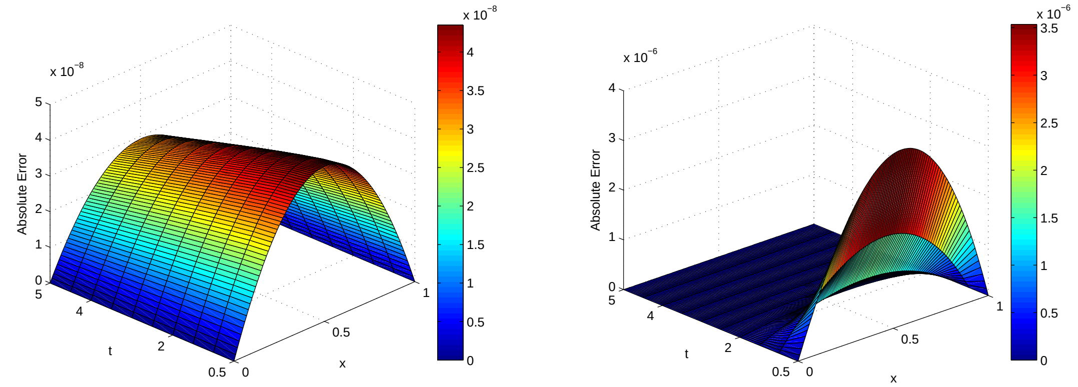
Table 18: Numerical solutions obtained using implicit exponential finite difference scheme with $h = 0.1$ and $k = 0.1$

		$\delta = 2, \alpha = 1, \beta = 5$			$\delta = 2, \alpha = 1, \beta = 0.5$		
t	x	Exact Solution	Numerical Solution	Absolute Error	Exact Solution	Numerical Solution	Absolute Error
1	0.1	0.999981	0.999912	$6.80309\text{x}10^{-5}$	0.872080	0.872097	$1.65765\text{x}10^{-5}$
	0.5	0.999975	0.999674	$3.00575\text{x}10^{-4}$	0.841820	0.842069	$2.49351\text{x}10^{-4}$
	0.9	0.999967	0.999865	$1.01782\text{x}10^{-4}$	0.806674	0.806692	$1.75363\text{x}10^{-5}$
10	0.1	1.000000	1.000000	0.0000	0.999997	0.999997	$2.48751\text{x}10^{-9}$
	0.5	1.000000	1.000000	0.0000	0.999997	0.999997	$3.34798\text{x}10^{-9}$
	0.9	1.000000	1.000000	0.0000	0.999996	0.999996	$4.23372\text{x}10^{-9}$

Table 19: Absolute error obtained using implicit exponential finite difference scheme with $\delta = 2, \alpha = \beta = 1$

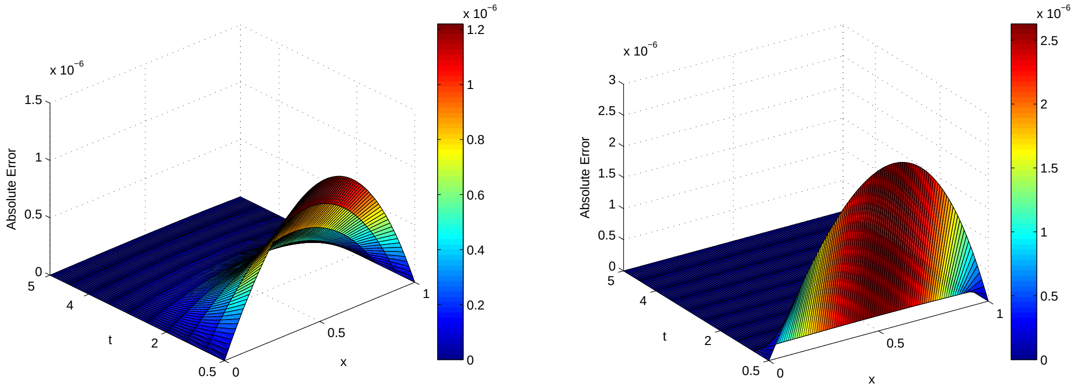
x	t	Absolute Error	[26]	[27]	[29]
0.1	0.0001	$6.91543\text{x}10^{-10}$	$2.80396\text{x}10^{-4}$	$1.17539\text{x}10^{-5}$	$1.08\text{x}10^{-6}$
	0.0005	$3.36024\text{x}10^{-9}$	$1.40177\text{x}10^{-3}$	$5.87633\text{x}10^{-5}$	$1.08\text{x}10^{-6}$
	0.001	$6.49282\text{x}10^{-9}$	$2.80301\text{x}10^{-3}$	$1.17512\text{x}10^{-4}$	$1.08\text{x}10^{-6}$
0.5	0.0001	$7.26918\text{x}10^{-10}$	$2.69094\text{x}10^{-4}$	$5.33686\text{x}10^{-5}$	$1.14\text{x}10^{-6}$
	0.0005	$3.49097\text{x}10^{-9}$	$1.34526\text{x}10^{-3}$	$1.06736\text{x}10^{-5}$	$1.14\text{x}10^{-6}$
	0.001	$6.93714\text{x}10^{-9}$	$2.69000\text{x}10^{-3}$	$1.06739\text{x}10^{-4}$	$1.14\text{x}10^{-6}$
0.9	0.0001	$1.21736\text{x}10^{-9}$	$2.55498\text{x}10^{-4}$	$9.29303\text{x}10^{-6}$	$4.12\text{x}10^{-6}$
	0.0005	$5.93745\text{x}10^{-9}$	$1.27699\text{x}10^{-3}$	$4.64718\text{x}10^{-5}$	$4.12\text{x}10^{-6}$
	0.001	$1.15244\text{x}10^{-8}$	$2.55346\text{x}10^{-4}$	$9.296\text{x}10^{-4}$	$4.12\text{x}10^{-6}$

Rate of convergence at $t = 1$ for $\delta = 1, \alpha = \beta = 0.001$ is shown in Table 20. From this table, it can be seen that errors approach to zero as the mesh refines, which shows that the scheme is consistent.



(a) $\alpha = \beta = 0.1$. (b) $\alpha = \beta = 1$.

Figure 11: Absolute errors at various time levels using present method for $\alpha = \beta = 0.1$ and $\alpha = \beta = 1$



(a) $\alpha = 1.0, \beta = 0.5$. (b) $\alpha = 1.0, \beta = 5.0$.

Figure 12: Absolute errors at various time levels using present method for $\alpha = 1$, $\beta = 0.5$ and $\alpha = 1, \beta = 5$

Table 20: Rate of convergence for $\delta = 1, \alpha = \beta = 0.001$ at $t = 1$

N	Rate
2	-
4	1.01385031
8	0.66289368
16	0.14849629
32	0.06067977
64	0.03530420

6. Conclusions

In this study, the Fisher's equation, the Burgers-Fisher equation and the generalized Burgers-Fisher equation are solved by implicit exponential finite difference method. As can be seen tables, the absolute errors are very small, therefore the present methods offer high accuracy for the numerical solutions of the equations. From comparison tables, the results obtained by the implicit exponential finite difference schemes are better than those obtained from the other numerical schemes such as Adomian Decomposition method, Exp-function method hybridized with Heuristic Computation and Optimal Homotopy Asymptotic method [26, 27, 29]. Also, it should be noted that the accuracy of the results decreased when δ and β increased and the accuracy increased when α decreased.

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