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Article

Classification of Surfaces of Coordinate Finite Type in the Lorentz–Minkowski 3-Space

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Abstract: In this paper, we define surfaces of revolution without parabolic points in three-dimensional Lorentz–Minkowski space. Then, we classify this class of surfaces under the condition $\Delta^{III}\boldsymbol{x} = A\boldsymbol{x}$, where Δ^{III} is the Laplace operator regarding the third fundamental form, and A is a real square matrix of order 3. We prove that such surfaces are either catenoids or surfaces of Enneper, or pseudo spheres or hyperbolic spaces centered at the origin.

Keywords: Laplace operator; surfaces in E^3_1; surfaces of revolution; surfaces of coordinate finite type

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1. Introduction

Euclidean immersions of finite type were introduced by B.-Y. Chen about thirty years ago, and it has been a topic of active research since then. Let M^n be an n-dimensional submanifold of an arbitrary dimensional Euclidean space E^m. Denote by Δ^I the Beltrami–Laplace operator on M^n with respect to the first fundamental form I of M^n. The submanifold M^n is said to be of finite *k-type* if its position vector $\boldsymbol{x}$ can be written as a sum of eigenvectors of the Laplace–Beltrami operator, Δ^I, according to k distinct eigenvalues, i.e., $\boldsymbol{x} = \mathbf{y}_0 + \mathbf{y}_1 + \cdots + \mathbf{y}_k$, for a constant vector $\mathbf{y}_0$ and smooth non-constant functions $\mathbf{y}_k$, $(i = 1, \ldots, k)$ such that $\Delta\mathbf{y}_i = \lambda_i\mathbf{y}_i$, $\lambda_i \in \mathbb{R}$, ref. [1].

In this respect, important families of surfaces were studied by different authors by proving that finite type ruled surfaces [2], finite type quadrics [3], finite type tubes [4], finite type cyclides of Dupin [5], and finite type spiral surfaces [6] are surfaces of the only known examples in E^3. However, for other classical families of surfaces, such as surfaces of revolution, translation surfaces as well as helicoidal surfaces, the classification of its finite type surfaces is not known yet. (For a survey in E^m, see [7]).

The year 1966 was the beginning when Takahashi in [8] stated that spheres and minimal surfaces are the only ones in E^3 whose position vector $\boldsymbol{x}$ satisfies the relation

$$\Delta^I\boldsymbol{x} = \lambda\boldsymbol{x}, \quad \lambda \in R. \tag{1}$$

Since the coordinate functions of $\boldsymbol{x}$ can be denoted as (x_1, x_2, x_3), then Takahashi's condition (1) becomes

$$\Delta^I x_i = \lambda x_i, \quad i = 1, 2, 3. \tag{2}$$

Later, in [9], Garay generalized Takahashi's condition (2). Actually, he studied surfaces of revolution in E^3, whose component functions satisfy the condition

$$\Delta^I x_i = \lambda_i x_i, \quad i = 1, 2, 3,$$

that is, the component functions are eigenfunctions of their Laplacian but not necessarily with the same eigenvalue. Another generalization was also made by studying surfaces whose position vector $\boldsymbol{x}$ satisfies a relation of the form

$$\Delta^I \boldsymbol{x} = A\boldsymbol{x},$$

where $A \in \mathbb{R}^{3\times 3}$ [10].

This type of study can also be extended to any smooth map, which is not necessary for the position vector of the surface, for example, the Gauss map of a surface. For the version of finite type Gauss map ruled surfaces, and tubes were studied in [11], while cyclides of Dupin were investigated in [12]. Concerning classes of surfaces whose Gauss map $\boldsymbol{n}$ satisfies $\Delta^I \boldsymbol{n} = A\boldsymbol{n}$, where $A \in \mathbb{R}^{3\times 3}$, one can find in [13] for the class of helicoidal surfaces, the class of tubular surfaces in [14], and, finally, the class of surfaces of revolution in [15].

Another extension can be drawn by applying the conditions stated before but for the 2nd or 3rd fundamental form of a surface [16]. More precisely, for the third fundamental form, ruled and quadric surfaces were studied in [17], translation surfaces were studied in [18], tubular surfaces in [19], and surfaces of revolution in [20]. The second fundamental form tubular surfaces were studied in [21], and surfaces of revolution were investigated in [22]. On the other hand, all the ideas mentioned above can be applied in the Lorentz–Minkowski space E^3_1.

Let M^2 be a connected non-degenerate submanifold in the three-dimensional Lorentz–Minkowski space E^3_1 and $\boldsymbol{x} : M^2 \to E^3_1$ be a parametric representation of a surface in the Lorentz–Minkowski 3-space E^3_1 equipped with the induced metric. Let (x, y, z) be a rectangular coordinate system of E^3_1. By saying Lorentz–Minkowski space E^3_1, we mean the Euclidean space E^3 with the standard metric given by

$$ds^2 = -dx^2 + dy^2 + dz^2.$$

Thus, an interesting geometric question has been posed: Classify all surfaces in E^3_1, which satisfy the condition

$$\Delta^J \boldsymbol{x} = A\boldsymbol{x}, \quad J = I, II, III, \tag{3}$$

where $A \in \mathbb{R}^{3\times 3}$ and Δ^J is the Laplace operator, regarding the fundamental form J.

Kaimakamis and Papantoniou in [23] solved the above question for the class of surfaces of revolution with respect to the second fundamental form. In [24], Bekkar and Zoubir studied the same class of surfaces with respect to the first fundamental form satisfying

$$\Delta x^i = \lambda^i x^i, \quad \lambda^i \in R.$$

Moreover, surfaces of revolution satisfying an equation according to the position vector field and the second Laplacian in E^3_1 were studied in [25]. Furthermore, coordinate finite-type submanifolds in pseudo-Euclidean spaces have been studied in [26,27]. An interesting piece of research one can also follow is the idea in [28] by defining the first and second Beltrami operator using the definition of the fractional vector operators.

In this paper, we investigate the Lorentz version of the surfaces of revolution satisfying the relation (3) according to the third fundamental form.

2. Basic Concepts

Let $C : \boldsymbol{r}(s) : s \in (a, b) \subset E \longrightarrow E^2$ be a curve in a plane E^2 of E^3_1 and l be a straight line of E^2, which does not intersect the curve C. A surface of revolution M^2 in E^3_1 is defined to be a non-degenerate surface, revolving the curve C around the axis l. If the axis l is space-like (resp. time-like), then l is transformed to the y-axis or z-axis (resp. x-axis) by the Lorentz transformation. Thus, we may consider the z-axis (resp. x-axis) as the axis l if it is space-like (resp. time-like). If the axis is null, then we may assume that this axis is the line spanned by the vector (1, 1, 0) of the xy-plane [23].

Firstly, we consider that the axis l is the z-axis (space-like) and the curve C is lying in the yz-plane or xz-plane. Then, C is parametrized as $\boldsymbol{r}(s) = (0, f(s), g(s))$ or $\boldsymbol{r}(s) = (f(s), 0, g(s))$, where f, g are smooth functions. Without loss of generality, we may assume that $f(s) > 0, s \in (a, b)$.

A subgroup of the Lorentz group which fixes the vector $(0,0,1)$ is given by [25]

$$\begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\theta \in \mathbb{R}$, (hyperbolic group). Therefore, the surface of revolution M^2 in E_1^3 in a system of local curvilinear coordinates (s, θ) is given by:

$$\boldsymbol{x}(s,\theta) = \big(f(s)\sinh\theta, f(s)\cosh\theta, g(s)\big) \tag{4}$$

or

$$\boldsymbol{x}(s,\theta) = \big(f(s)\cosh\theta, f(s)\sinh\theta, g(s)\big). \tag{5}$$

Secondly, let the axis l be the x-axis (time-like) lying in the xy-plane. Then, the curve C is given by $\boldsymbol{r}(s) = (g(s), f(s), 0)$, where $f(s) > 0, s \in (a, b)$. In this case, the subgroup of the Lorentz group which fixes the vector $(1,0,0)$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

where $\theta \in \mathbb{R}$ (elliptic group). Hence, the surface of revolution M^2 can be parametrized as

$$\boldsymbol{x}(s,\theta) = \big(g(s), f(s)\cos\theta, f(s)\sin\theta\big). \tag{6}$$

Finally, if the axis l is the line spanned by the vector (1, 1, 0), as the surface M^2 is non-degenerate, we can assume that the curve C lies in the xy-plane, i.e.,

$$\boldsymbol{r}(s) = (f(s), g(s), 0), \tag{7}$$

where $g = g(s)$ is a smooth positive function and $f = f(s)$ is a smooth function in the interval (a, b) such that $h(s) = f(s) - g(s) \neq 0$ for all $s \in (a, b)$. We notice here that the subgroup of the Lorentz group which fixes the vector $(1,1,0)$ consists of the matrix

$$\begin{bmatrix} 1+\frac{\theta^2}{2} & -\frac{\theta^2}{2} & \theta \\ \frac{\theta^2}{2} & 1-\frac{\theta^2}{2} & \theta \\ \theta & -\theta & 1 \end{bmatrix},$$

where $\theta \in \mathbb{R}$, (parabolic group). Hence, M^2 can be parametrized as

$$\boldsymbol{x}(s,\theta) = \big(f(s) + \frac{1}{2}\theta^2 h(s), g(s) + \frac{1}{2}\theta^2 h(s), \theta h(s)\big). \tag{8}$$

We denote by g_{km}, b_{km} and e_{km} with $k, m = 1, 2$ with the first, second and third fundamental forms of M^2, respectively, where we put

$$g_{11} = E =< \boldsymbol{x_s}, \boldsymbol{x_s} >, \quad g_{12} = F =< \boldsymbol{x_s}, \boldsymbol{x_\theta} >, \quad g_{22} = G =< \boldsymbol{x_\theta}, \boldsymbol{x_\theta} >,$$

$$b_{11} = L =< \boldsymbol{x_{ss}}, \boldsymbol{N} >, \quad b_{12} = M =< \boldsymbol{x_{s\theta}}, \boldsymbol{N} >, \quad b_{22} = N =< \boldsymbol{x_{\theta\theta}}, \boldsymbol{N} >,$$

$$e_{11} = \frac{EM^2 - 2FLM + GL^2}{EG - F^2} =< \boldsymbol{N_s}, \boldsymbol{N_s} >,$$
$$e_{12} = \frac{EMN - FLN + GLM - FM^2}{EG - F^2} =< \boldsymbol{N_s}, \boldsymbol{N_\theta} >,$$
$$e_{22} = \frac{GM^2 - 2FNM + EN^2}{EG - F^2} =< \boldsymbol{N_\theta}, \boldsymbol{N_\theta} >,$$

where $\boldsymbol{N}$ is the unit normal vector of M^2 and $<,>$ is the Lorentzian metric. For a sufficient differentiable function $p(u^1, u^2)$ on M^2, the second Laplace operator according to the fundamental form III of M^2 is defined by [29]:

$$\Delta^{III} p = -\frac{1}{\sqrt{e}} (\sqrt{e} e^{km} p_{/k})_{/m},$$

where $p_{/k} := \frac{\partial p}{\partial u^k}$, e^{km} denote the components of the inverse tensor of e_{km} and $e = \det(e_{km})$. After a long computation, we arrive at

$$\begin{aligned} \Delta^{III} p \quad = \quad & -\frac{\sqrt{\mid EG - F^2 \mid}}{LN - M^2} \Bigg(\Bigg(\frac{(GM^2 - 2FNM + EN^2)\frac{\partial p}{\partial s}}{(LN - M^2)\sqrt{\mid EG - F^2 \mid}} \\ & - \frac{(EMN - FLN + GLM - FM^2)\frac{\partial p}{\partial \theta}}{(LN - M^2)\sqrt{\mid EG - F^2 \mid}} \Bigg)_s \\ & - \Bigg(\frac{(EMN - FLN + GLM - FM^2)\frac{\partial p}{\partial s}}{(LN - M^2)\sqrt{\mid EG - F^2 \mid}} - \frac{(EM^2 - 2FLM + GL^2)\frac{\partial p}{\partial \theta}}{(LN - M^2)\sqrt{\mid EG - F^2 \mid}} \Bigg)_\theta \Bigg). \end{aligned} \tag{9}$$

Here, we have $LN - M^2 \neq 0$, since the surface has no parabolic points.

3. Proof of the Main Results

In this paragraph, we classify the surfaces of revolution M^2 satisfying the relation (3). We distinguish the following three types according to whether these surfaces are determined.

3.1. Type I

The parametric representation of M^2 is given by (4) with a space-like axis. Suppose that $\boldsymbol{r}$ is parametrized by arc-length, that is, it satisfies

$$f'^2(s) + g'^2(s) = 1. \tag{10}$$

By considering this with (4), we obtain that the components of the first fundamental form are

$$E = 1, \quad F = 0, \quad G = -f^2, \tag{11}$$

and also by using (4) and the unit normal vector $\boldsymbol{N}$ of M^2, we have the components of the second fundamental form

$$L = -f'g'' + g'f'', \quad M = 0, \quad N = fg'. \tag{12}$$

Denote by κ the curvature of the curve C and r_1, r_2 the principal radii of curvature of M^2. We have

$$r_1 = \kappa, \quad r_2 = \frac{g'}{f},$$

and

$$K = r_1 r_2 = \frac{\kappa g'}{f} = -\frac{f''}{f}, \quad 2H = r_1 + r_2 = \kappa + \frac{g'}{f}, \tag{13}$$

which are the Gaussian curvature and the mean curvature of M^2, respectively. Since the relation (10) holds, there exists a smooth function $\varphi = \varphi(s)$ such that

$$f' = \cos\varphi, \quad g' = \sin\varphi. \tag{14}$$

Then, $\kappa = \varphi'$ and relations (12), (13) become

$$L = -\varphi', \quad M = 0, \quad N = f\sin\varphi, \tag{15}$$

$$K = \frac{\varphi'\sin\varphi}{f} \quad \text{and} \quad 2H = -\varphi' - \frac{\sin\varphi}{f}. \tag{16}$$

We put $r = \frac{1}{r_1} + \frac{1}{r_2} = \frac{2H}{K}$. Thus, we have

$$r = -\left(\frac{1}{\varphi'} + \frac{f}{\sin\varphi}\right). \tag{17}$$

Taking the derivative of the last equation, we obtain

$$r' = \frac{\varphi''}{\varphi'^2} + \frac{f\varphi'\cos\varphi}{\sin^2\varphi} - \frac{\cos\varphi}{\sin\varphi}. \tag{18}$$

From (9), (11), and (15), we have

$$\Delta^{III}\boldsymbol{x} = -\frac{1}{\varphi'^2}\frac{\partial^2\boldsymbol{x}}{\partial s^2} + \frac{1}{\sin^2\varphi}\frac{\partial^2\boldsymbol{x}}{\partial\theta^2} + \left(\frac{\varphi''}{\varphi'^3} - \frac{\cos\varphi}{\varphi'\sin\varphi}\right)\frac{\partial\boldsymbol{x}}{\partial s}. \tag{19}$$

Let (x_1, x_2, x_3) be the coordinate functions of the position vector $\boldsymbol{x}$ of (4). Then, according to relations (2), (19) and taking into account (17) and (18), we find that

$$\Delta^{III}x_1 = \Delta^{III}f(s)\sinh\theta = \left(-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'}\right)\sinh\theta, \tag{20}$$

$$\Delta^{III}x_2 = \Delta^{III}f(s)\cosh\theta = \left(-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'}\right)\cosh\theta, \tag{21}$$

$$\Delta^{III}x_3 = \Delta^{III}g(s) = r\cos\varphi + r'\frac{\sin\varphi}{\varphi'}. \tag{22}$$

We denote by $a_{ij}, i,j = 1,2,3$, the entries of the matrix A, where all entries are real numbers. By using (20)–(22), condition (3) is found to be equivalent to the following system:

$$\left(-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'}\right)\sinh\theta = a_{11}f(s)\sinh\theta + a_{12}f(s)\cosh\theta + a_{13}g(s), \tag{23}$$

$$\left(-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'}\right)\cosh\theta = a_{21}f(s)\sinh\theta + a_{22}f(s)\cosh\theta + a_{23}g(s), \tag{24}$$

$$r\cos\varphi + r'\frac{\sin\varphi}{\varphi'} = a_{31}f(s)\cosh\theta + a_{32}f(s)\sinh\theta + a_{33}g(s). \tag{25}$$

From (25), it can be easily verified that $a_{31} = a_{32} = 0$. On the other hand, differentiating (23) and (24) twice with respect to θ, we obtain that $a_{13} = a_{23} = 0$. Thus, the system is reduced to

$$\left(-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'}\right)\sinh\theta = a_{11}f(s)\sinh\theta + a_{12}f(s)\cosh\theta, \tag{26}$$

$$\left(-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'}\right)\cosh\theta = a_{21}f(s)\sinh\theta + a_{22}f(s)\cosh\theta, \tag{27}$$

$$r\cos\varphi + r'\frac{\sin\varphi}{\varphi'} = a_{33}g(s). \tag{28}$$

However, $\sinh\theta$ and $\cosh\theta$ are linearly independent functions of θ, so we deduce that $a_{12} = a_{21} = 0$ and $a_{11} = a_{22}$. Putting $a_{11} = a_{22} = \lambda$ and $a_{33} = \mu$, we see that the system of Equations (26)–(28) reduces now to the following two equations:

$$-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'} = \lambda f, \tag{29}$$

$$r\cos\varphi + r'\frac{\sin\varphi}{\varphi'} = \mu g. \tag{30}$$

Hence, the matrix A for which relation (3) is satisfied becomes

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Solving the system (29) and (30) with respect to r and r', we conclude that

$$r' = \varphi'(\lambda f\cos\varphi + \mu g\sin\varphi), \tag{31}$$

$$r = \mu g\cos\varphi - \lambda f\sin\varphi. \tag{32}$$

Taking the derivative of (32), we find

$$r' = \frac{1}{2}(\mu - \lambda)\cos\varphi\sin\varphi. \tag{33}$$

We distinguish now the following cases:

Case I. $\mu = \lambda = 0$. In this case, from (32), we have $r = 0$. Consequently, by considering (16) and (17), we conclude $H = 0$. That is, M^2 is minimal.

Case II. $\mu = \lambda \neq 0$. Then, from (33), we obtain $r' = 0$. Now, by considering this into (31), we discuss two cases. First, if $\varphi' = 0$, then the surface M^2 would consist only of parabolic points, which has been excluded. Therefore, we left with

$$f(s)\cos\varphi + g(s)\sin\varphi = 0,$$

or by considering (14)

$$ff' + gg' = 0,$$

from which we obtain $f^2 + g^2 = c^2, c \in R$. Therefore, the surface M^2 obviously satisfies the equation $-x^2 + y^2 + z^2 = c^2$, that is, M^2 is an open piece of the pseudo-sphere $S_1^2(0,c)$ centered at the origin with radius c on E_1^3.

Case III. $\lambda \neq 0, \mu = 0$. Then, system (29), (30) is equivalently reduced to

$$-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'} = \lambda f(s),$$

$$r\cos\varphi + r'\frac{\sin\varphi}{\varphi'} = 0.$$

From (32), we have

$$r + \lambda f\sin\varphi = 0. \tag{34}$$

On differentiating (34) and taking into account (31) with $\mu = 0$, we obtain

$$\lambda f\varphi'\cos\varphi + \lambda\cos\varphi\sin\varphi + \lambda f\varphi'\cos\varphi = 0$$

or

$$\varphi' = -\frac{\sin\varphi}{2f}.$$

From (34), (17) and the last equation, we obtain

$$\frac{f}{\sin\varphi} + \lambda f \sin\varphi = 0$$

or

$$f(1 + \lambda \sin^2\varphi) = 0.$$

It is a contradiction. Hence, there are no surfaces of revolution with parametric representation (4) of E_1^3 satisfying (3).

Case IV. $\lambda = 0, \mu \neq 0$. Then, Equations (29) and (30) reduced to

$$-r\sin\varphi + r'\frac{\cos\varphi}{\varphi'} = 0,$$

$$r\cos\varphi + r'\frac{\sin\varphi}{\varphi'} = \mu g. \tag{35}$$

From (32), we have

$$r - \mu g\cos\varphi = 0. \tag{36}$$

Taking the derivative of (36) and taking into account (31) with $\lambda = 0$, we find

$$\mu g\varphi'\sin\varphi - \mu\cos\varphi\sin\varphi + \mu g\varphi'\sin\varphi = 0$$

or

$$\varphi' = \frac{\cos\varphi}{2g}. \tag{37}$$

Taking the derivative of (37), we find

$$3\varphi'\sin\varphi + 2g\varphi'' = 0. \tag{38}$$

On account of (35), (17) and (18), it is easily verified that

$$\varphi'' = \frac{\varphi'^2}{\sin\varphi}(\mu g\varphi' + 2\cos\varphi). \tag{39}$$

Inserting (37) and (39) in (38), we conclude

$$3 + (\frac{1}{2}\mu - 1)\cos^2\varphi = 0.$$

Here, we also have a contradiction.

Case V. $\lambda \neq 0, \mu \neq 0$. We write Equations (29) and (30) as follows:

$$\frac{\sin\varphi}{\varphi'} + \frac{f}{\sin^2\varphi} + \frac{\varphi''\cos\varphi}{\varphi'^3} - \frac{\cos^2\varphi}{\varphi'\sin\varphi} - \lambda f = 0, \tag{40}$$

$$\frac{\varphi''\sin\varphi}{\varphi'^3} - \frac{2\cos\varphi}{\varphi'} - \mu g = 0. \tag{41}$$

From (41), we have relation (39). By eliminating φ'' from (40), we obtain

$$\frac{1}{\varphi'\sin\varphi} + \frac{\mu g\cos\varphi}{\sin\varphi} + \frac{f}{\sin^2\varphi} - \lambda f = 0. \tag{42}$$

On differentiating the last equation and using (39), we find

$$\frac{2\mu g\varphi'}{\sin^2\varphi} + \frac{2f\varphi'\cos\varphi}{\sin^3\varphi} + \frac{2\cos\varphi}{\sin^2\varphi} - (\mu-\lambda)\cos\varphi = 0. \tag{43}$$

Multiplying (42) by $\frac{2\varphi'}{\sin\varphi}$ and (43) by $-\cos\varphi$, we obtain

$$\frac{2}{\sin^2\varphi} + \frac{2\mu g\varphi'\cos\varphi}{\sin^2\varphi} + \frac{2f\varphi'}{\sin^3\varphi} - \frac{2\lambda\varphi' f}{\sin\varphi} = 0, \tag{44}$$

$$-\frac{2\mu g\varphi'\cos\varphi}{\sin^2\varphi} - \frac{2f\varphi'\cos^2\varphi}{\sin^3\varphi} - \frac{2\cos^2\varphi}{\sin^2\varphi} + (\mu-\lambda)\cos^2\varphi = 0. \tag{45}$$

Combining (44) and (45), we conclude that

$$(\mu-\lambda)\cos^2\varphi - 2(\lambda-1)\frac{f\varphi'}{\sin\varphi} + 2 = 0 \tag{46}$$

or

$$\frac{(\mu-\lambda)\cos^2\varphi}{\varphi'} - 2(\lambda-1)\frac{f}{\sin\varphi} + \frac{2}{\varphi'} = 0.$$

Taking the derivative of the above equation and using (33) and (39), we find

$$2(\mu+1)\cos\varphi + (\mu-\lambda)\mu g\varphi'\cos^2\varphi - 2(\lambda-1)\frac{f\varphi'\cos\varphi}{\sin\varphi} + 2\mu g\varphi' = 0. \tag{47}$$

Multiplying (46) by $-\cos\varphi$, and adding the resulting equation to (47), we obtain

$$2\mu\cos\varphi + (2+(\mu-\lambda)\cos^2\varphi)\mu g\varphi' - (\mu-\lambda)\cos^3\varphi = 0$$

or

$$2\mu\cos^2\varphi + (2+(\mu-\lambda)\cos^2\varphi)\mu g\varphi'\cos\varphi - (\mu-\lambda)\cos^4\varphi = 0. \tag{48}$$

On account of (42), we find

$$\mu g\varphi'\cos\varphi = \lambda f\varphi'\sin\varphi - \frac{f\varphi'}{\sin\varphi} - 1. \tag{49}$$

Eliminating $\mu g\varphi'\cos\varphi$ from (48) by using (49), Equation (48) reduces to

$$\begin{aligned}&2\mu\cos^2\varphi - (\mu-\lambda)\cos^4\varphi + \\ &\big(2+(\mu-\lambda)\cos^2\varphi\big)\Big((\lambda\sin^2\varphi - 1)\frac{f\varphi'}{\sin\varphi} - 1\Big) = 0.\end{aligned} \tag{50}$$

However, from (46), we have

$$\frac{f\varphi'}{\sin\varphi} = \frac{(\mu-\lambda)\cos^2\varphi + 2}{2(\lambda-1)}. \tag{51}$$

Obviously $\lambda \neq 1$ because otherwise, from (46), we would have

$$(\mu-\lambda)\cos^2\varphi + 2 = 0.$$

This is a contradiction. Now, by inserting (51) in (50), we obtain

$$\begin{aligned}&-\lambda(\mu-\lambda)^2\cos^4\varphi + (\mu-\lambda)\big((\mu-\lambda)(\lambda-1) - 6\lambda + 2\big)\cos^2\varphi \\ &+6\mu(\lambda-1) - 2\lambda(\lambda+1) = 0.\end{aligned}$$

This relation, however, is valid for a finite number of values of φ. Thus, in this case, there are no surfaces of revolution with the required property.

Now, let us consider a surface of revolution M^2 given by (5). Suppose that $\boldsymbol{r}$ is parametrized by arc-length, that is, it satisfies

$$g'^2(s) - f'^2(s) = -\varepsilon, \ (\varepsilon = \pm 1).$$

Here, also, one can find

$$\begin{aligned} E = -\varepsilon, \quad F = 0, \quad G = f^2, \\ L = f'g'' - f''g', \quad M = 0, \quad N = -g'f. \end{aligned}$$

By using the same procedure as above, we have the following:

If $\varepsilon = 1$, M^2 is an open piece of the pseudo-sphere $\mathbb{S}_1^2(0,c)$ centered at the origin with radius c, or minimal surface.

If $\varepsilon = -1$, M^2 is an open piece of the hyperbolic space $\mathbb{H}_1^2(0,c)$ centered at the origin with radius c, or minimal surface. Thus, we proved the following:

Theorem 1. *Let $\boldsymbol{x} : M^2 \longrightarrow E_1^3$ be a surface of revolution with a space-like axis. Then, $\boldsymbol{x}$ satisfies (3) regarding to the third fundamental form if and only if one of the following statements holds:*

- *M^2 has zero mean curvature;*
- *M^2 is an open piece of the pseudo-sphere $\mathbb{S}_1^2(0,c)$ centered at the origin with radius c;*
- *M^2 is an open piece of the hyperbolic space $\mathbb{H}_1^2(0,c)$ centered at the origin with radius c.*

3.2. Type II

The parametric representation of M^2 is given by (6) with a time-like axis. Then, the tangent vector of the profile curve parametrized by arc-length is

$$< \boldsymbol{x}', \boldsymbol{x}' >= f'^2 - g'^2 = \varepsilon, \ (\varepsilon = \pm 1).$$

We can assume that

$$f'^2 - g'^2 = 1, \quad \forall s \in (a,b). \tag{52}$$

Then, the components of the first and second fundamental forms are given by, respectively,

$$E = 1, \quad F = 0, \quad G = f^2, \tag{53}$$

and

$$L = f'g'' - g'f'', \quad M = 0, \quad N = fg'. \tag{54}$$

From Equation (52), it is obviously clear that there exists a smooth function $\varphi = \varphi(s)$ such that

$$f' = \cosh\varphi, \quad g' = \sinh\varphi.$$

On the other hand, similar to the way followed in the previous type, we can obtain

$$r_1 = \kappa = \varphi', \quad r_2 = \frac{g'}{f} = \frac{\sinh\varphi}{f},$$

and so the Gaussian curvature and mean curvature are given by

$$K = r_1 r_2 = \frac{\kappa g'}{f} = -\frac{f''}{f} = \frac{\varphi' \sinh\varphi}{f}, \quad 2H = r_1 + r_2 = \varphi' + \frac{\sinh\varphi}{f}. \tag{55}$$

Here, we have

$$r = \frac{1}{\varphi'} + \frac{f}{\sinh\varphi}. \tag{56}$$

By taking the derivative of the last equation, we obtain

$$r' = -\frac{\varphi''}{\varphi'^2} - \frac{f\varphi' \cosh\varphi}{\sinh^2\varphi} + \frac{\cosh\varphi}{\sinh\varphi}. \tag{57}$$

On the other hand, by considering (53) and (54) in (9), we obtain

$$\Delta^{III}\boldsymbol{x} = -\frac{1}{\varphi'^2}\frac{\partial^2 \boldsymbol{x}}{\partial s^2} - \frac{1}{\sinh^2\varphi}\frac{\partial^2 \boldsymbol{x}}{\partial \theta^2} + \left(\frac{\varphi''}{\varphi'^3} - \frac{\cosh\varphi}{\varphi'\sinh\varphi}\right)\frac{\partial \boldsymbol{x}}{\partial s}. \tag{58}$$

By substituting the components of (6) into (58), we find

$$\Delta^{III}x_1 = \Delta^{III}g(s) = -r\cosh\varphi - r'\frac{\sinh\varphi}{\varphi'},$$

$$\Delta^{III}x_2 = \Delta^{III}f(s)\cos\theta = \left(-r\sinh\varphi - r'\frac{\cosh\varphi}{\varphi'}\right)\cos\theta,$$

$$\Delta^{III}x_3 = \Delta^{III}f(s)\sin\theta = \left(-r\sinh\varphi - r'\frac{\cosh\varphi}{\varphi'}\right)\sin\theta.$$

Now let $\Delta^{III}\boldsymbol{x} = A\boldsymbol{x}$. Thus, as in the former paragraph, we find

$$-r\cosh\varphi - r'\frac{\sinh\varphi}{\varphi'} = a_{11}g(s) + a_{12}f(s)\cos\theta + a_{13}f(s)\sin\theta,$$

$$\left(-r\sinh\varphi - r'\frac{\cosh\varphi}{\varphi'}\right)\cos\theta = a_{21}g(s) + a_{22}f(s)\cos\theta + a_{23}f(s)\sin\theta,$$

$$\left(-r\sinh\varphi - r'\frac{\cosh\varphi}{\varphi'}\right)\sin\theta = a_{31}g(s) + a_{32}f(s)\cos\theta + a_{33}f(s)\sin\theta.$$

Applying the same algebraic methods, used in the previous type, the above system reduces to

$$-r\cosh\varphi - r'\frac{\sinh\varphi}{\varphi'} = \mu g, \tag{59}$$

$$-r\sinh\varphi - r'\frac{\cosh\varphi}{\varphi'} = \lambda f, \tag{60}$$

where $a_{11} = \mu, a_{22} = a_{33} = \lambda$, $\lambda, \mu \in R$. Solving the system (59) and (60) with respect to r and r', we conclude that

$$r' = \varphi'(-\lambda f\cosh\varphi + \mu g\sinh\varphi), \tag{61}$$

$$r = \lambda f\sinh\varphi - \mu g\cosh\varphi. \tag{62}$$

Now, we consider the following five cases according to the values of λ, μ.

Case I. $\lambda = \mu = 0$. Thus, from (62), we conclude that $r = 0$. Consequently, by considering (55) and (56), we conclude that $H = 0$. That is, M^2 is minimal.

Case II. $\mu = \lambda \neq 0$. Then, from (61), we have that $r' = 0$. If $\varphi' = 0$, then M^2 would consist only of parabolic points, which has been excluded. Therefore, we find that

$$-f\cosh\varphi + g\sinh\varphi = 0$$

or

$$-ff' + gg' = 0.$$

Then, $g^2 - f^2 = c^2, c \in R$ and, therefore, M^2 is obviously the hyperbolic space $H^2(0, c)$ centered at the origin with an imaginary radius, given by $x^2 + y^2 - z^2 = -c^2$.

Case III. $\lambda \neq 0, \mu = 0$. Then, system (59), (60) is reduced to

$$r\cosh\varphi + r'\frac{\sinh\varphi}{\varphi'} = 0,$$

$$-r\sinh\varphi - r'\frac{\cosh\varphi}{\varphi'} = \lambda f(s).$$

From (62), we obtain

$$r - \lambda f\sinh\varphi = 0. \tag{63}$$

By differentiating (63) and taking into account (61) with $\mu = 0$, we obtain

$$\varphi' = -\frac{\sinh\varphi}{2f}.$$

Considering (56), (62) and the last equation together, we obtain

$$f(1 - \lambda\sin^2\varphi) = 0,$$

which is a contradiction. Hence, there are no surfaces of revolution with parametric representation (6) of E_1^3 satisfying (3).

Case IV. $\lambda = 0, \mu \neq 0$. Then, Equations (59) and (60) reduced to

$$-r\cosh\varphi - r'\frac{\sinh\varphi}{\varphi'} = \mu g, \tag{64}$$

$$-r\sinh\varphi - r'\frac{\cosh\varphi}{\varphi'} = 0.$$

From (62), we have

$$r + \mu g\cosh\varphi = 0. \tag{65}$$

Taking the derivative of (65) and taking into account (61) with $\lambda = 0$, we find

$$\varphi' = -\frac{\cosh\varphi}{2g}. \tag{66}$$

Taking the derivative of (66), we obtain

$$3\varphi'\sinh\varphi + 2g\varphi'' = 0. \tag{67}$$

On account of (56), (57) and (64), it is easily verified that

$$\varphi'' = \frac{\varphi'^2}{\sinh\varphi}(\mu g\varphi' + 2\cosh\varphi). \tag{68}$$

Inserting (66) and (68) in (67), we conclude that

$$3 - (\frac{1}{2}\mu + 1)\cosh^2\varphi = 0,$$

which shows that it is a contradiction.

Case V. Let $\lambda \neq 0, \mu \neq 0$. Now, by substituting (56) and (57) into Equations (59) and (60), we can rewrite this system as

$$\frac{\varphi''\sinh\varphi}{\varphi'^3} - \frac{2\cosh\varphi}{\varphi'} - \mu g = 0, \tag{69}$$

$$-\frac{\sinh\varphi}{\varphi'} + \frac{f}{\sinh^2\varphi} + \frac{\varphi''\cosh\varphi}{\varphi'^3} - \frac{\cosh^2\varphi}{\varphi'\sinh\varphi} - \lambda f = 0. \tag{70}$$

From (69), we have relation (68). By eliminating φ'' from (70), we obtain

$$\frac{1}{\varphi' \sinh\varphi} + \frac{\mu g \cosh\varphi}{\sinh\varphi} + \frac{f}{\sinh^2\varphi} - \lambda f = 0. \tag{71}$$

On differentiating the last equation and using (68), we find

$$\frac{2\mu g\varphi'}{\sinh^2\varphi} + \frac{2f\varphi'\cosh\varphi}{\sinh^3\varphi} + \frac{2\cosh\varphi}{\sinh^2\varphi} - (\mu-\lambda)\cosh\varphi = 0. \tag{72}$$

Multiplying (71) by $\frac{2\varphi'}{\sinh\varphi}$ and (72) by $-\cosh\varphi$, we obtain

$$\frac{2}{\sinh^2\varphi} + \frac{2\mu g\varphi'\cosh\varphi}{\sinh^2\varphi} + \frac{2f\varphi'}{\sinh^3\varphi} - \frac{2\lambda\varphi' f}{\sinh\varphi} = 0, \tag{73}$$

$$-\frac{2\mu g\varphi'\cosh\varphi}{\sinh^2\varphi} - \frac{2f\varphi'\cosh^2\varphi}{\sinh^3\varphi} - \frac{2\cosh^2\varphi}{\sinh^2\varphi} + (\mu-\lambda)\cosh^2\varphi = 0. \tag{74}$$

Combining (73) and (74), we conclude that

$$(\mu-\lambda)\cosh^2\varphi - 2(\lambda+1)\frac{f\varphi'}{\sinh\varphi} - 2 = 0 \tag{75}$$

or

$$\frac{(\mu-\lambda)\cosh^2\varphi}{\varphi'} - 2(\lambda+1)\frac{f}{\sinh\varphi} - \frac{2}{\varphi'} = 0.$$

Taking the derivative of the above equation and using (68), we find

$$2(\mu+1)\cosh\varphi + (\mu-\lambda)\mu g\varphi'\cosh^2\varphi - 2(\lambda+1)\frac{f\varphi'\cosh\varphi}{\sinh\varphi} - 2\mu g\varphi' = 0. \tag{76}$$

Multiplying (75) by $-\cosh\varphi$, and adding the resulting equation to (76), we obtain

$$2(\mu+2)\cosh\varphi - (2-(\mu-\lambda)\cos^2\varphi)\mu g\varphi' - (\mu-\lambda)\cosh^3\varphi = 0$$

or

$$2(\mu+2)\cosh^2\varphi - (2-(\mu-\lambda)\cosh^2\varphi)\mu g\varphi'\cos\varphi - (\mu-\lambda)\cos^4\varphi = 0. \tag{77}$$

On account of (71), we find

$$\mu g\varphi'\cosh\varphi = \lambda f\varphi'\sinh\varphi - \frac{f\varphi'}{\sinh\varphi} - 1. \tag{78}$$

Eliminating $\mu g\varphi'\cosh\varphi$ from (77) by using (78), we obtain

$$\begin{aligned} &2(\mu+2)\cosh^2\varphi - (\mu-\lambda)\cosh^4\varphi - \\ &(2-(\mu-\lambda)\cosh^2\varphi)\Big((\lambda\sinh^2\varphi - 1)\frac{f\varphi'}{\sinh\varphi} - 1\Big) = 0. \end{aligned} \tag{79}$$

However, from (75), we have

$$\frac{f\varphi'}{\sinh\varphi} = \frac{2-(\mu-\lambda)\cosh^2\varphi}{2(\lambda+1)}. \tag{80}$$

Obviously, $\lambda \neq -1$ because, otherwise, from (75), we would have

$$(\mu-\lambda)\cosh^2\varphi - 2 = 0.$$

This is a contradiction. Now, by inserting (80) in (79), we obtain

$$-\lambda(\mu-\lambda)^2\cos^6\varphi+(\mu-\lambda)\big((\mu-\lambda)(\lambda-1)+4\lambda\big)\cos^4\varphi$$
$$+(6\lambda^2-2\lambda-2\mu-2\lambda\mu+8)\cos^2\varphi+8(\lambda+1)=0.$$

This relation, however, is valid for a finite number of values of φ. Thus, in this case, there are no surfaces of revolution with the required property.

Finally, let $\varepsilon=1$, i.e., M^2 is a time-like surface. Quite similarly as before, we can show that M^2 is an open part of the pseudo-sphere $S_1^2(0,c)$ centered at the origin with real radius c, given by the equation $x^2+y^2-z^2=c^2$, or minimal, or the catenoid of the 3rd kind as a time-like surface. Thus, we proved the following:

Theorem 2. *Let $\boldsymbol{x}: M^2 \longrightarrow E_1^3$ be a surface of revolution given by (6). Then, $\boldsymbol{x}$ satisfies (3) regarding to the third fundamental form if and only if one of the following statements holds:*

- *M^2 has zero mean curvature;*
- *M^2 is an open piece of the pseudo-sphere $\mathbb{S}_1^2(0,c)$ centered at the origin with real radius c;*
- *M^2 is an open piece of the hyperbolic space $\mathbb{H}_1^2(0,c)$ centered at the origin with real radius c.*

3.3. Type III

The parametric representation of M^2 is given by (8), i.e.,

$$\boldsymbol{x}(s,\theta)=\big(f(s)+\frac{1}{2}\theta^2h(s),g(s)+\frac{1}{2}\theta^2h(s),\theta h(s)\big),$$

where $h(s)=f(s)-g(s)\neq 0$. Since M^2 is non-degenerate, $f'(s)^2-g'(s)^2$ never vanishes, and so $h'(s)=f'(s)-g'(s)\neq 0$ everywhere. Now, we may take the parameter in such a way that

$$h(s)=-2s.$$

Assume that $k(s)=g(s)-s$; then,

$$f(s)=k(s)-s \quad g(s)=k(s)+s,$$

(see, for example, ref. [30]). Therefore, M^2 can be reparametrized as follows:

$$\boldsymbol{x}(s,\theta)=\big(k-s-\theta^2 s,k+s-\theta^2 s,-2s\theta\big), \tag{81}$$

with the profile curve given in (7) becomes

$$\boldsymbol{r}(s)=(0,k(s)-s,k(s)+s). \tag{82}$$

By using the tangent vector fields, $\boldsymbol{x}_s$ and $\boldsymbol{x}_\theta$ of M^2, the components of the first and second fundamental forms are given by

$$E=4k'(s),\quad F=0,\quad G=4s^2.$$

Now, let M^2 be a space-like surface, i.e., $k'(s)>0$. Then, the time-like unit normal vector field $\boldsymbol{N}$ of M^2 is given by

$$\boldsymbol{N}=\frac{1}{2\sqrt{k'}}(\theta^2+1,\theta^2-1,2\theta)+\frac{\sqrt{k'}}{2}(1,1,0). \tag{83}$$

Then, the components of the second fundamental forms are given by

$$L=-\frac{k''}{\sqrt{k'}},\quad M=0,\quad N=\frac{2s}{\sqrt{k'}}.$$

Thus, relation (9) becomes

$$\Delta^{III} p = -\frac{4k'^2}{k''^2}\frac{\partial^2 p}{\partial s^2} - k'\frac{\partial^2 p}{\partial \theta^2} + \frac{2k'}{k''^3}\left(2k'k''' - k''^2\right)\frac{\partial p}{\partial s}. \quad (84)$$

According to relations (8) and (84), we find that

$$\Delta^{III} x_1 = \Delta^{III}(k - s - s\theta^2) = \frac{2k'}{k''^3}\left(2k'k''' - k''^2\right)(k' - 1 - \theta^2) - \frac{4k'^2}{k''} + 2sk',$$

$$\Delta^{III} x_2 = \Delta^{III}(k + s - s\theta^2) = \frac{2k'}{k''^3}\left(2k'k''' - k''^2\right)(k' + 1 - \theta^2) - \frac{4k'^2}{k''} + 2sk',$$

$$\Delta^{III} x_3 = \Delta^{III}(-2s\theta) = -\frac{4k'}{k''^3}\left(2k'k''' - k''^2\right)\theta.$$

Now, let $\Delta^{III}\boldsymbol{x} = A\boldsymbol{x}$. Then,

$$\frac{2k'}{k''^3}\left(2k'k''' - k''^2\right)(k' - 1 - \theta^2) - \frac{4k'^2}{k''} + 2sk' = a_{11}(k - s - s\theta^2) + a_{12}(k + s - s\theta^2) + a_{13}(-2s\theta), \quad (85)$$

$$\frac{2k'}{k''^3}\left(2k'k''' - k''^2\right)(k' + 1 - \theta^2) - \frac{4k'^2}{k''} + 2sk' = a_{21}(k - s - s\theta^2) + a_{22}(k + s - s\theta^2) + a_{23}(-2s\theta), \quad (86)$$

$$-\frac{4k'}{k''^3}\left(2k'k''' - k''^2\right)\theta = a_{31}(k - s - s\theta^2) + a_{32}(k + s - s\theta^2) + a_{33}(-2s\theta). \quad (87)$$

Regarding the above equations as polynomials in θ, so from the coefficients of (87), we obtain

$$(a_{31} + a_{32})s = 0, \quad (88)$$

$$\frac{2k'}{k''^3}\left(2k'k''' - k''^2\right) = a_{33}s, \quad (89)$$

$$(a_{32} - a_{31})s + (a_{31} + a_{32})k = 0. \quad (90)$$

From the coefficients of (86), we find

$$\frac{2k'}{k''^3}\left(2k'k''' - k''^2\right) = (a_{21} + a_{22})s, \quad (91)$$

$$a_{23}s = 0, \quad (92)$$

$$\frac{2k'}{k''^3}\left(2k'k''' - k''^2\right)(k' + 1) - \frac{4k'^2}{k''} + 2sk' = (a_{21} + a_{22})k + (a_{22} - a_{21})s. \quad (93)$$

From the coefficients of (85), we obtain

$$\frac{2k'}{k''^3}\left(2k'k''' - k''^2\right) = (a_{11} + a_{12})s, \quad (94)$$

$$a_{13}s = 0, \quad (95)$$

$$\frac{2k'}{k''^3}\left(2k'k''' - k''^2\right)(k' - 1) - \frac{4k'^2}{k''} + 2sk' = (a_{11} + a_{12})k + (a_{12} - a_{11})s. \quad (96)$$

It is easily verified that

$$a_{23} = a_{31} = a_{32} = a_{13} = 0.$$

On the other hand, from (89), (91) and (94), we find

$$a_{11} + a_{12} = a_{33} = a_{21} + a_{22},$$

from which we obtain

$$a_{12} = a_{33} - a_{11}, \quad a_{21} = a_{33} - a_{22}. \tag{97}$$

Moreover, by considering (89) and (97) in (93) and (96), respectively, we obtain

$$a_{33}s(k' + 1) - \frac{4k'^2}{k''} + 2sk' = a_{33}k + (a_{22} - a_{21})s \tag{98}$$

and

$$a_{33}s(k' - 1) - \frac{4k'^2}{k''} + 2sk' = a_{33}k + (a_{12} - a_{11})s. \tag{99}$$

By subtracting (98) from (99), we obtain

$$(a_{11} - a_{21})s + (a_{22} - a_{12})s - 2a_{33}s = 0. \tag{100}$$

From (97) and (100), we find

$$a_{21} = -a_{12}. \tag{101}$$

Taking into account relations (100) and (101), we obtain

$$a_{11} + a_{22} = 2a_{33}.$$

We put $a_{11} = \lambda$ and $a_{22} = \mu$, so the matrix A for which relation (3) is satisfied takes finally the following form:

$$A = \begin{bmatrix} \lambda & \frac{1}{2}(\mu - \lambda) & 0 \\ \frac{1}{2}(\lambda - \mu) & \mu & 0 \\ 0 & 0 & \frac{1}{2}(\lambda + \mu) \end{bmatrix}.$$

Hence, the system of Equations (88)–(96) reduces to the following two equations:

$$\frac{2k'}{k''^3}(2k'k''' - k''^2) = a_{33}s, \tag{102}$$

$$(a_{33} + 2)k's + 2a_{12}s - \frac{4k'^2}{k''} - a_{33}k = 0, \tag{103}$$

where, as we mentioned before, $a_{33} = \frac{1}{2}(\lambda + \mu)$ and $a_{12} = \frac{1}{2}(\mu - \lambda)$.

Solving the system of Equations (102) and (103) with respect to λ and μ, we find

$$\lambda = \frac{k'(2s - k + sk')}{s^2k''}\left(\frac{2k'k'''}{k''^2} - 1\right) - \frac{2k'^2}{sk''} + k', \tag{104}$$

$$\mu = \frac{k'(2s + k - sk')}{s^2k''}\left(\frac{2k'k'''}{k''^2} - 1\right) + \frac{2k'^2}{sk''} - k'. \tag{105}$$

Case I. $\lambda = \mu = 0$. Thus, from (104) and (105), we conclude that $k = as^3 + b$ with $a > 0$, b is a constant, and $s \neq 0$. Consequently, $H = 0$. Therefore, M^2 is minimal and the corresponding matrix A is the zero matrix.

Case II. $\lambda = \mu \neq 0$. Thus, from *Case I*, $k \neq as^3 + b$. Now, from (67), we obtain $a_{23} = 0$, and so

$$\frac{(k - sk')(2k'k''' - k''^2)}{s^2k''^3} + \frac{2k'}{sk''} - 1 = 0, \tag{106}$$

whose solution is $k(s) = \pm\frac{c^2}{4s}$. By considering (82), we conclude that $\boldsymbol{r}$ is a spherical curve and so the surface M^2 is an open piece of the pseudo-sphere $\mathbb{S}^2_1(0,c)$ or the hyperbolic space $\mathbb{H}^2(0,c)$.

Case III. $\lambda \neq 0, \mu = 0$. By considering the last assumption in (105), i.e., $\mu = 0$, we have

$$\frac{2k'}{sk''}\Big(\frac{2k'k'''}{k''^2} - 1\Big) = \frac{k'(-k+sk')}{s^2k''}\Big(\frac{2k'k'''}{k''^2} - 1\Big) - \frac{2k'^2}{sk''} + k'.$$

By substituting this into (104), we obtain

$$\lambda = \frac{4k'}{sk''}\Big(\frac{2k'k'''}{k''^2} - 1\Big),$$

where λ is a non-zero function. Since there is no k function to implement in both conditions, there is no surface of revolution that fulfills these conditions.

Case IV. $\lambda = 0, \mu \neq 0$. Similarly, we obtain a contradiction as in *Case III.*

Case V. $\lambda \neq \mu$ and $\lambda \neq 0, \mu \neq 0$. In this case, the above two relations (104) and (105) are valid only when λ and μ are functions of s. Thus, there are no surfaces of revolution with the required property. Thus, we proved the following:

Theorem 3. *Let* $\boldsymbol{x} : M^2 \longrightarrow E^3_1$ *be a surface of revolution given by (8). Then,* $\boldsymbol{x}$ *satisfies (3) regarding to the third fundamental form if and only if the following statements hold true:*
- M^2 *has zero mean curvature;*
- M^2 *is an open piece of the pseudo sphere* $\mathbb{S}^2_1(0,c)$ *of real radius* c*;*
- M^2 *is an open piece of the hyperbolic space* $\mathbb{H}^2_1(0,c)$ *of real radius* c*.*

Finally, we know that the minimal surfaces of revolution with a non-light-like axis are congruent to a part of the catenoid and also with a light-like axis are congruent to a part of the surface of Enneper (see for more details [31]). Now, by combining Theorem 1–3, and [31]:

Theorem 4. *(Classification) Let* $\boldsymbol{x} : M^2 \longrightarrow E^3_1$ *be a surface of revolution satisfying (3) regarding the third fundamental form. Then, M is one of the following:*
- M^2 *is an open part of catenoid of the 1st kind, the 2nd kind, the 3rd kind, the 4th kind, or the 5th kind.*
- M^2 *is an open part of the surface of Enneper of the 2nd kind or the 3rd kind,*
- M^2 *is an open part of the pseudo sphere* $\mathbb{S}^2_1(0,c)$ *centered at the origin with radius* c*,*
- M^2 *is an open part of the hyperbolic space* $\mathbb{H}^2_1(0,c)$ *centered at the origin with radius* c*.*

4. Discussion

Firstly, we introduce the class of surfaces of revolution of the 1st, 2nd, and 3rd kind as space-like or time-like in the Lorentz–Minkwoski 3-space. Then, we define a formula for the Laplace operator regarding the third fundamental form III. Finally, we classify the surfaces of revolution M^2 satisfying the relation $\Delta^{III}\boldsymbol{x} = A\boldsymbol{x}$, for a real square matrix A of order 3. We distinguish three types according to whether these surfaces are determined, with each type investigated in a subsection of Section 3. An interesting study can be drawn, if this type of study can be applied to other classes of surfaces that have not been investigated yet such as spiral surfaces, quadric surfaces, or tubular surfaces.

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