Alexandria Engineering Journal (2022) 61, 2651–2661

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Alexandria Engineering Journal

www.elsevier.com/locate/aej
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A numerical method for solving a class of systems of nonlinear Pantograph differential equations

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Received 11 May 2021; revised 27 June 2021; accepted 18 July 2021
Available online 15 September 2021

KEYWORDS

The systems of nonlinear Pantograph differential equation;
Collocation method;
Fibonacci polynomials

Abstract In this paper, Fibonacci collocation method is firstly used for approximately solving a class of systems of nonlinear Pantograph differential equations with initial conditions. The problem is firstly reduced into a nonlinear algebraic system via collocation points, later the unknown coefficients of the approximate solution function are calculated. Also, some problems are presented to test the performance of the proposed method by using the absolute error functions. Additionally, the obtained numerical results are compared with exact solutions of the test problems and approximate ones obtained with other methods in the literature.

1. Introduction

Many problems in science and engineering can be modeled with fractional and integer order partial differential equations, ordinary differential equations, integro-differential equations and their systems. For this reason, there are many studies on these equations in the literature. Some of these studies can be summarized as follows: In [1–4], the authors study on existence of fractional integro-differential equations. The paper [5] deals with the systems of fractional order Willis aneurysm and nonlinear singularly perturbed boundary value problems. The paper [6] is on the controllability results of the non-dense Hilfer neutral fractional derivative. In [7], on exact controllability of a class of fractional neutral integro-differential systems is studied. In [8], it is obtained exact solitary wave solutions of the strain wave equation. In [9,10], it is studied on Nizhnik-Novikov-Vesselov equations and modified Veronese web equation. In [11], it is presented solutions of nonlinear rth dispersionless equation. The paper [12] deals with solutions for the Kawahara-KdV type equations. The paper [13] is on new soliton solutions for a generalized nonlinear evolution equation. In [14], the authors study on the existence of mild solution of functional integro differential equation. The paper [15] is on abundant solitary wave solutions to an extended nonlinear Schrödinger's equation with conformable derivative. In [16], on the existence of Sobolev-type Hilfer fractional neutral integro-differential systems are studied using measure of noncompactness. In [17], on the existence and uniqueness of nonlocal fractional delay differential systems are studied. The paper [18] is on the approximate controllability of Hilfer fractional neutral stochastic integro-differential equations. The paper [19] deals with the existence and controllability results

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Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2021.07.028
1110-0168

for fractional evolution. In [20], the authors study on the existence and controllability of fractional integro-differential system. In [21], on a class of control systems governed by the fractional differential evolution equations is considered.

Solving systems of nonlinear Pantograph differential equation is highly important because of their role in the modeling of scientific phenomena and engineering. Due to the difficulties on obtaining the analytical solutions, several numerical methods are developed to solve those equations approximately. Some of the applied numerical methods on the approximate solutions of systems of nonlinear Pantograph differential equation are as follows: Variational iteration method [22], operational matrix method based on Bernoulli polynomials [23], optimized decomposition method [24], homotopy analysis method [25].

In [26], the Fibonacci collocation method is applied to linear differential-difference equations. Similarly, in [27],the high-order linear Fredholm integro-differential-difference equations are used by using the Fibonacci collocation method. In [28], a class of systems of linear Fredholm integro-differential equations is studied by the method. The paper given by [29] deals with that the application of the Fibonacci collocation method to singularly perturbed differential-difference equations. Also, in [30], the Fibonacci collocation method is used for approximately solving a class of systems of high-order linear Volterra integro-differential equations.

In this paper, the Fibonacci collocation method is developed for solving the following class of systems of nonlinear Pantograph differential equation that is very useful while modeling natural systems:[31,32]

$$\sum_{k=0}^{2}\sum_{r=1}^{2}P_{jkr}(x)u_r^{(k)}(x)+\sum_{k=0}^{2}\sum_{r=1}^{2}Q_{j\alpha_{js}\beta_{jp}}(x)u_s^r(\alpha_{js}x)u_p^{(k)}(\beta_{jp}x)=g_j(x) \tag{1.1}$$

$0 \leqslant x \leqslant 1,\ j,s,p=1,2$

with the initial conditions

$$\sum_{k=0}^{1}\left[a_{jk}u_r^{(k)}(0)+b_{jk}u_r^{(k)}(0)\right]=\delta_{jr}, \qquad j=1,2 \tag{1.2}$$

where $u_r^{(0)}(x)=u_r(x),\ u_r^0(x)=1$ and $u_r(x)$ is an unknown functions. $P_{jkr}(x), Q_{j\alpha_{js}\beta_{jp}}$ and $g_j(x)$are given continuous functions on interval $[0,1], a_{jk}, b_{jk}, \alpha_{js}, \beta_{jp}$and $\delta_{jr}$are suitable constants. The aim of this study is to get the approximate solutions as the truncated Fibonacci series defined by

$$u_r(x)=\sum_{n=1}^{N+1}c_{rn}F_n(x) \tag{1.3}$$

where $F_n(x)$denotes the Fibonacci polynomials; c_{rn} $(1 \leqslant rn \leqslant N+1)$are unknown Fibonacci polynomial coefficients, and N is chosen as any positive integer such that $N \geqslant m$.

The paper consists of six sections. In Section 2, the basic properties and definitions related to Fibonacci polynomials are presented. In Section 3, the fundamental matrix forms of the Fibonacci collocation method by using fundamental relations of Fibonacci polynomials are constructed to obtain the approximate solutions for the given class of systems of nonlinear Pantograph differential equation. In Section 4, the absolute error function is formulated. In Section 5, three test problems are presented and the method is tested using the absolute error function. Finally, conclusions are given in Section 6.

2. Properties of Fibonacci polynomials

The *Fibonacci polynomials* were studied by Falcon and Plaza [33,34]. The recurrence relation of those polynomials is defined by

$$F_n(x)=xF_{n-1}(x)+F_{n-2}(x) \tag{2.1}$$

For $n \geqslant 3. F_1(x)=1,\ F_2(x)=x$. The properties were further investigated by Falcon and Plaza in [33,34]. The first few Fibonacci polynomials are

$$\begin{aligned}
F_1 &= \mathbf{1}\\
F_2 &= 1x\\
F_3 &= \mathbf{1}x^2+\mathbf{1}\\
F_4 &= 1x^3+2x\\
F_5 &= \mathbf{1}x^4+\mathbf{3}x^2+\mathbf{1}\\
F_6 &= 1x^5+4x^3+3x\\
F_7 &= \mathbf{1}x^6+\mathbf{5}x^4+\mathbf{6}x^2+\mathbf{1}\\
F_8 &= 1x^7+6x^5+10x^3+4x\\
F_9 &= \mathbf{1}x^8+\mathbf{7}x^6+\mathbf{15}x^4+\mathbf{10}x^2+\mathbf{1}\\
F_{10} &= 1x^9+8x^7+21x^5+20x^3+5x\\
&\ \vdots
\end{aligned} \tag{2.2}$$

The compact form of Eq. (2.2) is given by

$$F_n(x)=\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n-i-1}{i}x^{n-2i-1}, \quad \left[\frac{n-1}{2}\right]=\begin{cases}\frac{n-2}{2}, & n\ even,\\ \frac{n-1}{2}, & n\ odd.\end{cases}$$

Besides, a relationship between the Fibonacci polynomials and the standard basis polynomials is given by

$$x^n=F_{n+1}(x)+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\left[\binom{n}{k}-\binom{n}{k-1}\right]F_{n+1-2k}(x).$$

For example,

$$\begin{aligned}
x^0 &= F_1(x)\\
x^1 &= F_2(x)\\
x^2 &= F_3(x)-F_1(x)\\
x^3 &= F_4(x)-2F_2(x)\\
x^4 &= F_5(x)-3F_3(x)+2F_1(x)\\
x^5 &= F_6(x)-4F_4(x)+5F_2(x)\\
x^6 &= F_7(x)-5F_5(x)+9F_3(x)-5F_1(x)\\
x^7 &= F_8(x)-6F_6(x)+14F_4(x)-14F_2(x)
\end{aligned} \tag{2.3}$$

3. Fundamental relations

Let us assume that linear combination of Fibonacci polynomials (1.3) is an approximate solutions of Eq. (1.1). Our purpose is to determine the matrix forms of Eq. (1.1) by using (1.3).

Firstly, we can write Fibonacci polynomials (2.2) in the matrix form

$$\mathbf{F}(x) = \mathbf{T}(x)\mathbf{M} \tag{3.1}$$

where $\mathbf{F}(x) = [F_1(x)\ F_2(x)\cdots F_{N+1}(x)], \mathbf{T}(x) = [1\ x\ x^2\ x^3 \ldots x^N]$, $\mathbf{C}_r = [c_{r1}\ c_{r2}\ \cdots c_{r(N+1)}]^T, r = 1, 2$and

$$\mathbf{M} = \begin{bmatrix} \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \cdots \\ 0 & 1 & 0 & 2 & 0 & 3 & 0 & 4 & 0 & 5 & \cdots \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{3} & 0 & \mathbf{6} & 0 & \mathbf{10} & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 10 & 0 & 20 & \cdots \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{5} & 0 & \mathbf{15} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 & 0 & 21 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{7} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 8 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then we set the approximate solutions defined by a truncated Fibonacci series (1.3) in the matrix form

$$u_r(x) = \mathbf{F}(x)\mathbf{C}_r. \tag{3.2}$$

By using the relations (3.1) and (3.2), the matrix relation is expressed as

$$\begin{aligned} u_r(x) &\cong u_{rN}(x) = \mathbf{P}(x)\mathbf{C}_r = \mathbf{T}(x)\mathbf{M}\mathbf{C}_r \\ u'_r(x) &\cong u'_{rN}(x) = \mathbf{T}\mathbf{B}\mathbf{M}\mathbf{C}_r \\ u''_r(x) &\cong u''_{rN}(x) = \mathbf{T}(x)\mathbf{B}^2\mathbf{M}\mathbf{C}_r \\ &\ldots \\ u_r^{(k)}(x) &\cong u_{rN}^{(k)}(x) = \mathbf{T}(x)\mathbf{B}^k\mathbf{M}\mathbf{C}_r \end{aligned} \tag{3.3}$$

where $r = 1, 2$. Also, the relations between the matrix $\mathbf{T}(x)$and its derivatives, $\mathbf{T}'(x), \mathbf{T}''(x),\ldots,\mathbf{T}^{(k)}(x)$are

$$\begin{aligned} \mathbf{T}'(x) &= \mathbf{T}(x)\mathbf{B}, \mathbf{T}''(x) = \mathbf{T}(x)\mathbf{B}^2 \\ \mathbf{T}'''(x) &= \mathbf{T}(x)\mathbf{B}^3, \ldots, \mathbf{T}^{(k)}(x) = \mathbf{T}(x)\mathbf{B}^k \end{aligned} \tag{3.4}$$

Then we set the approximate solution defined by a truncated Fibonacci series (1.3) in the matrix form

$$u_r(x) \cong u_{rN}(x) = \mathbf{F}(x)\mathbf{C}_r. \tag{3.5}$$

By substituting the Fibonacci collocation points defined by

$$x_i = \frac{i}{N}, \quad i = 0, 1, \ldots N \tag{3.6}$$

into Eq. (3.3), we have

$$u_r^{(k)}(x_i) = \mathbf{T}(x_i)\mathbf{B}^k\mathbf{M}\mathbf{C}_r \tag{3.7}$$

and the compact form of the relation (3.7) becomes

$$\mathbf{U}_r^{(k)} = \mathbf{T}\mathbf{B}^k\mathbf{M}\mathbf{C}_r, \quad k = 0, 1, 2, \qquad r = 1, 2 \tag{3.8}$$

where

$$\mathbf{U}_r^{(k)} = \begin{bmatrix} u_r^{(k)}(x_0) \\ u_r^{(k)}(x_1) \\ \vdots \\ u_r^{(k)}(x_N) \end{bmatrix}, \tag{3.9}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\mathbf{B}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}(x_0) \\ \mathbf{T}(x_1) \\ \vdots \\ \mathbf{T}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \ldots & x_0^N \\ 1 & x_1 & \ldots & x_1^N \\ 1 & \vdots & \ldots & \vdots \\ 1 & x_N & \ldots & x_N^N \end{bmatrix}.$$

In addition, we can obtain the matrix form $\left(\hat{\mathbf{U}}_{s,\alpha_{js}}\right)^r \hat{\mathbf{U}}_{p,\beta_{jp}}^{(k)}$ which appears in the nonlinear part of Eq. (1.1), by using Eq. (3.3) as

$$\left(\hat{\mathbf{U}}_{s,\alpha_{js}}\right)^r \hat{\mathbf{U}}_{p,\beta_{jp}}^{(k)} = \begin{bmatrix} u_s^r(\alpha_{js}x_0)u_p^{(k)}(\beta_{jp}x_0) \\ u_s^r(\alpha_{js}x_1)u_p^{(k)}(\beta_{jp}x_1) \\ \vdots \\ u_s^r(\alpha_{js}x_N)u_p^{(k)}(\beta_{jp}x_N) \end{bmatrix} \tag{3.10}$$

$$= \begin{bmatrix} u_s(\alpha_{js}x_0) & 0 & \ldots & 0 \\ 0 & u_s(\alpha_{js}x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_s(\alpha_{js}x_N) \end{bmatrix}^r \begin{bmatrix} u_p^{(k)}(\beta_{jp}x_0) \\ u_p^{(k)}(\beta_{jp}x_1) \\ \vdots \\ u_p^{(k)}(\beta_{jp}x_N) \end{bmatrix} \tag{3.11}$$

where

$$\left(\hat{\mathbf{U}}_{s,\alpha_{js}}\right)^r \hat{\mathbf{U}}_{p,\beta_{jp}}^{(k)} = \left(\hat{\mathbf{T}}_{s,\alpha_{js}}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_r\right)^r \mathbf{T}_{p,\beta_{jp}}\ (\mathbf{B})^{\mathbf{k}}\mathbf{M}. \tag{3.12}$$

$$\hat{\mathbf{T}}_{s,\alpha_{js}} = \begin{bmatrix} \mathbf{T}(\alpha_{js}x_0) & 0 & \ldots & 0 \\ 0 & \mathbf{T}(\alpha_{js}x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{T}(\alpha_{js}x_N) \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \ldots & 0 \\ 0 & \mathbf{B} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{B} \end{bmatrix},$$

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & 0 & \ldots & 0 \\ 0 & \mathbf{M} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{M} \end{bmatrix}, \quad \hat{\mathbf{C}}_r = \begin{bmatrix} \mathbf{C}_r & 0 & \ldots & 0 \\ 0 & \mathbf{C}_r & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{C}_r \end{bmatrix}.$$

Substituting the collocation points $(x_i = i/N, i = 0, 1, \cdots, N)$ into Eq. (1.1), gives the system of equations

$$\sum_{k=0}^{2}\sum_{r=1}^{2} P_{jkr}(x_i)u_r^{(k)}(x_i) + \sum_{k=0}^{2}\sum_{r=1}^{2} Q_{jkr\alpha\beta}(x_i)u_s^r(\alpha_{js}x_i)u_p^{(k)}(\beta_{jp}x_i) = g_j(x_i), \quad 0 \leqslant x \leqslant 1$$

which can be expressed with the aid of Eqs. (3.9) and (3.10) as

$$\sum_{k=0}^{2}\sum_{r=1}^{2} P_{jkr}\mathbf{U}_r^{(k)} + \sum_{k=0}^{2}\sum_{r=1}^{2} Q_{jkr\alpha\beta}\left(\hat{\mathbf{U}}_{s,\alpha_{js}}\right)^r \hat{\mathbf{U}}_{p,\beta_{jp}}^{(k)} = \mathbf{G}_j \tag{3.13}$$

where

$$P_{jkr} = \; diag\left[P_{jkr}(x_0) \;\; P_{jkr}(x_1) \;\; \ldots \;\; P_{jkr}(x_N)\right],$$
$$Q_{jkr\alpha\beta} = \; diag\left[Q_{jkr\alpha\beta}(x_0) \;\; Q_{jkr\alpha\beta}(x_1) \;\; \ldots \;\; Q_{jkr\alpha\beta}(x_N)\right]$$

and

$$\mathbf{G}_j = \left[gj(x_0) \quad g_j(x_1) \quad \ldots \quad g_j(x_N)\right]^T, \; j = 1, 2.$$

Substituting the relations (3.8) and (3.12) into Eq. (3.13), the fundamental matrix equation can be obtained as

$$\left\{\sum_{k=0}^{2}\sum_{r=1}^{2} P_{jkr}\mathbf{T}\mathbf{B}^k\mathbf{M} + \sum_{k=0}^{2}\sum_{r=1}^{2} Q_{jkr\alpha\beta}\left(\hat{\mathbf{T}}_{s,\alpha_{js}}\hat{\mathbf{M}}\;\hat{\mathbf{C}}_r\right)^r \mathbf{T}_{p,\beta_{jp}}(\mathbf{B})^{\mathbf{k}}\mathbf{M}\right\}\mathbf{C}_r = \mathbf{G}_j \tag{3.14}$$

Briefly, Eq. (3.14) can also be written in the form,

$$\mathbf{W}\mathbf{C} = \mathbf{G} \;\; \text{or} \;\; [\mathbf{W}; \mathbf{G}] \tag{3.15}$$

where

$$\mathbf{W} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \; \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}, \; \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}$$

$$W_{11} = \sum_{k=0}^{2}\sum_{r=1}^{1} P_{jkr}\mathbf{T}\mathbf{B}^k\mathbf{M} + \sum_{k=0}^{2}\sum_{r=1}^{1} Q_{jkr\alpha\beta}\left(\hat{\mathbf{T}}_{s,\alpha_{js}}\hat{\mathbf{M}}\;\hat{\mathbf{C}}_r\right)^r \mathbf{T}_{p,\beta_{jp}}(\mathbf{B})^{\mathbf{k}}\mathbf{M} \quad \textbf{for} \;\; \mathbf{j = 1}$$

$$W_{12} = \sum_{k=0}^{2}\sum_{r=2}^{2} P_{jkr}\mathbf{T}\mathbf{B}^k\mathbf{M} + \sum_{k=0}^{2}\sum_{r=2}^{2} Q_{jkr\alpha\beta}\left(\hat{\mathbf{T}}_{s,\alpha_{js}}\hat{\mathbf{M}}\;\hat{\mathbf{C}}_r\right)^r \mathbf{T}_{p,\beta_{jp}}(\mathbf{B})^{\mathbf{k}}\mathbf{M} \quad \textbf{for} \;\; \mathbf{j = 1}$$

$$W_{21} = \sum_{k=0}^{2}\sum_{r=1}^{1} P_{jkr}\mathbf{T}\mathbf{B}^k\mathbf{M} + \sum_{k=0}^{2}\sum_{r=1}^{1} Q_{jkr\alpha\beta}\left(\hat{\mathbf{T}}_{s,\alpha_{js}}\hat{\mathbf{M}}\;\hat{\mathbf{C}}_r\right)^r \mathbf{T}_{p,\beta_{jp}}(\mathbf{B})^{\mathbf{k}}\mathbf{M} \quad \textbf{for} \;\; \mathbf{j = 2}$$

$$W_{12} = \sum_{k=0}^{2}\sum_{r=2}^{2} P_{jkr}\mathbf{T}\mathbf{B}^k\mathbf{M} + \sum_{k=0}^{2}\sum_{r=2}^{2} Q_{jkr\alpha\beta}\left(\hat{\mathbf{T}}_{s,\alpha_{js}}\hat{\mathbf{M}}\;\hat{\mathbf{C}}_r\right)^r \mathbf{T}_{p,\beta_{jp}}(\mathbf{B})^{\mathbf{k}}\mathbf{M} \quad \textbf{for} \;\; \mathbf{j = 2.}$$

Here, Eq. (3.15) corresponds to a system of the $(N+1)$ nonlinear algebraic equations with the unknown Fibonacci coefficients $c_{rn}, \;\; n = 1, 2, \ldots, N+1$.

Now, a matrix representation of the mixed conditions in Eq. (1.2) can be found. Using the relation in Eq. (3.3) at points 0 and 1, the matrix representation of the mixed conditions in Eq. (1.2) that depends on the Fibonacci coefficients in matrix $\mathbf{C}_r$ becomes

$$\left\{\sum_{k=0}^{m-1}\left[a_{jk}\mathbf{T}(0) + b_{jk}\mathbf{T}(0)\right](\mathbf{B})^{(k)}\mathbf{M}\right\}\mathbf{C}_r = \delta_{jr}, \quad j = 0, 1, 2, \ldots, m-1$$

or briefly

$$\mathbf{V}_{jr}\;\mathbf{C}_r = \left[\delta_{jr}\right] \quad \text{or} \; \left[\mathbf{V}_{jr}; \delta_{jr}\right]; \quad j = 0, 1, 2, \ldots, m-1 \tag{3.16}$$

where

$$\mathbf{V}_{jr} = \sum_{k=0}^{m-1}\left[a_{jk}\mathbf{T}(0) + b_{jk}\mathbf{T}(0)\right](\mathbf{B})^{(k)}\mathbf{M} = \left[v_{jo} \quad v_{j1} \quad v_{j2} \ldots \; v_{jN}\right].$$

Consequently, by replacing the row matrices in (3.16) by the m rows of the augmented matrix (3.15), the new augmented matrix becomes

$$\hat{\mathbf{W}}\;\mathbf{C} = \hat{\mathbf{G}} \quad \text{or} \;\; \left[\hat{\mathbf{W}}; \hat{\mathbf{G}}\right]$$

which is a nonlinear algebraic system. For convenience, if the last rows of the matrix are replaced, the new augmented matrix of the above system is as follows:

$$\left[\hat{\mathbf{W}}; \hat{\mathbf{G}}\right] = \begin{bmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{21} & \hat{W}_{22} \end{bmatrix} \tag{3.17}$$

where

$$\left[\hat{W}_{11}\right] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1N+1} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2N+1} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(N+1-m)1} & w_{(N+1-m)2} & w_{(N+1-m)3} & \cdots & w_{(N+1-m)N+1} \\ v_{11} & v_{12} & v_{13} & \cdots & v_{1N+1} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2N+1} \\ v_{31} & v_{32} & v_{33} & \cdots & v_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{(m-1)1} & v_{(m-1)2} & v_{(m-1)3} & \cdots & v_{(m-1)N+1} \end{bmatrix}$$

$$\left[\hat{W}_{12}\right] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1N+1} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2N+1} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(N+1-m)1} & w_{(N+1-m)2} & w_{(N+1-m)3} & \cdots & w_{(N+1-m)N+1} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\left[\hat{W}_{21}\right] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1N+1} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2N+1} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(N+1-m)1} & w_{(N+1-m)2} & w_{(N+1-m)3} & \cdots & w_{(N+1-m)N+1} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\left[\hat{W}_{22}\right] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1N+1} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2N+1} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(N+1-m)1} & w_{(N+1-m)2} & w_{(N+1-m)3} & \cdots & w_{(N+1-m)N+1} \\ v_{11} & v_{12} & v_{13} & \cdots & v_{1N+1} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2N+1} \\ v_{31} & v_{32} & v_{33} & \cdots & v_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{(m-1)1} & v_{(m-1)2} & v_{(m-1)3} & \cdots & v_{(m-1)N+1} \end{bmatrix}$$

$$\hat{\mathbf{G}} = \begin{bmatrix} \hat{\mathbf{G}}_1 \\ \hat{\mathbf{G}}_2 \end{bmatrix}$$

where

$$\hat{\mathbf{G}}_1 = [g_1(x_0) \quad g_1(x_1) \quad \cdots \quad g_1(x_{N+1-m}) \quad \delta_{10} \quad \delta_{11} \quad \delta_{12} \quad \cdots \quad \delta_{1m-1}]^T$$
$$\hat{\mathbf{G}}_2 = [g_2(x_0) \quad g_2(x_1) \quad \cdots \quad g_2(x_{N+1-m}) \quad \delta_{20} \quad \delta_{21} \quad \delta_{22} \quad \cdots \quad \delta_{2m-1}]^T.$$

However, the last rows in the above matrix do not need to be replaced. For example, if matrix $\hat{\mathbf{W}}$ is singular, the rows that have the same factor or are all zero are replaced. Thus, by solving the linear equation system in (3.17), the unknown Fibonacci coefficients c_{rn}, $n = 1, 2, \ldots, N+1$ are determined and substituted into (1.3), and the Fibonacci polynomial solutions is found.

with initial conditions

$$u_1(0) = -3, \quad u_2(0) = -2, u_1'(0) = 0$$

and the exact solutions $u_1(x) = x^2 - 3$, $u_2(x) = x - 2$. The approximate the solution $u_r(x)$ by the Fibonacci polynomials is

$$u_r(x) = \sum_{n=1}^{N+1} c_{rn} F_n(x)$$

where

$$N = 2,\ P_{121}(x) = 1,\ P_{111}(x) = 1, P_{112}(x) = -x, P_{101}(x) = 1, (Q_{111}(x) = 1,\ \alpha_{11} = 1,\ \beta_{12} = 1),$$
$$\left(Q_{11\frac{1}{2}}(x) = 1,\ \alpha_{11} = 1,\ \beta_{12} = \tfrac{1}{2}\right), g_1(x) = \tfrac{5x^3}{2} - 3x^2 - \tfrac{7x}{2} + 11, \text{and}$$
$$P_{221}(x) = x,\ P_{212}(x) = 1,\ P_{201}(x) = 1,\ (Q_{211}(x) = 1,\ \alpha_{21} = 1,\ \beta_{21} = 1),$$
$$\left(Q_{2\frac{11}{55}}(x) = -1,\ \alpha_{22} = \tfrac{1}{5},\ \beta_{22} = \tfrac{1}{5}\right),\quad g_2(x) = x^4 - \tfrac{126x^2}{25} + \tfrac{14x}{5} + 3.$$

4. Error estimation

In this section, to test the accuracy of the proposed method, it is presented that the absolute error functions $E_{1,N}(x)$ and $E_{2,N}(x)$. The functions $E_{1,N}(x)$ and $E_{2,N}(x)$ are given by

$$E_{1,N}(x) = |u_{1,N}(x) - u_1(x)| \tag{4.1}$$

and

$$E_{2,N}(x) = |u_{2,N}(x) - u_2(x)| \tag{4.2}$$

where $u_{1,N}(x)$ and $u_{2,N}(x)$ are the approximate solutions of Eq. (1.1) according to N. Besides, $u_1(x)$ and $u_2(x)$ are the exact solutions of Eq. (1.1).

5. Numerical examples

In this section, three numerical examples are presented to illustrate the efficiency of the proposed method. On these problems, the method is tested by using the error functions given by (4.1) and (4.2). The obtained numerical results are presented with tables and graphics.

Example 1. Consider the second order nonlinear differential equation

$$\begin{cases} u_1''(x) + u_1'(x) - xu_2'(x) + u_1(x) + u_1(x)u_2(x) + u_1(x)u_2\left(\frac{x}{2}\right) = \frac{5x^3}{2} - 3x^2 - \frac{7x}{2} + 11 \\ xu_1''(x) + u_2'(x) + u_1(x) + u_1^2(x) - u_2^2\left(\frac{x}{5}\right) = x^4 - \frac{126x^2}{25} + \frac{14x}{5} + 3 \end{cases} \tag{5.1}$$

Hence, the set of collocation points (3.6) for $N = 2$ is computed as

$$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1$$

From Eq. (3.14), the fundamental matrix equation of the problem is

$$\mathbf{G_1} = \left\{\mathbf{P}_{121}\mathbf{TB}^2\mathbf{M} + \mathbf{P}_{111}\mathbf{TBM} + \mathbf{P}_{101}\mathbf{TM} + \mathbf{Q}_{111}\hat{\mathbf{T}}_{1,1}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_2\mathbf{T}_{1,1}\ \mathbf{M}\right\}\mathbf{C}_1$$
$$+\left\{\mathbf{P}_{112}\mathbf{TBM} + \mathbf{Q}_{11\frac{1}{2}}\hat{\mathbf{T}}_{2,1}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_2\mathbf{T}_{2,\frac{1}{2}}\mathbf{M}\right\}\mathbf{C}_2$$
$$\mathbf{G_2} = \left\{\mathbf{P}_{221}\mathbf{TB}^2\mathbf{M} + \mathbf{P}_{201}\mathbf{TM} + \mathbf{Q}_{211}\hat{\mathbf{T}}_{1,1}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_1\mathbf{T}_{1,1}\mathbf{M}\right\}\mathbf{C}_1$$
$$+\left\{\mathbf{P}_{212}\mathbf{TBM} + \mathbf{Q}_{2\frac{11}{55}}\hat{\mathbf{T}}_{2,\frac{1}{5}}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_2\mathbf{T}_{2,\frac{1}{5}}\mathbf{M}\right\}\mathbf{C}_2$$

where

$$\mathbf{W}_{11} = \mathbf{P}_{121}\mathbf{TB}^2\mathbf{M} + \mathbf{P}_{111}\mathbf{TBM} + \mathbf{P}_{101}\mathbf{TM} + \mathbf{Q}_{111}\hat{\mathbf{T}}_{1,1}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_2\mathbf{T}_{1,1}\mathbf{M}$$
$$\mathbf{W}_{12} = \mathbf{P}_{112}\mathbf{TBM} + \mathbf{Q}_{11\frac{1}{2}}\hat{\mathbf{T}}_{2,1}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_2\mathbf{T}_{2,\frac{1}{2}}\mathbf{M}$$
$$\mathbf{W}_{21} = \mathbf{P}_{221}\mathbf{TB}^2\mathbf{M} + \mathbf{P}_{201}\mathbf{TM} + \mathbf{Q}_{211}\hat{\mathbf{T}}_{1,1}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_1\mathbf{T}_{1,1}\mathbf{M}$$
$$\mathbf{W}_{22} = \mathbf{P}_{212}\mathbf{TBM} + \mathbf{Q}_{2\frac{11}{55}}\hat{\mathbf{T}}_{2,\frac{1}{5}}\hat{\mathbf{M}}\ \hat{\mathbf{C}}_2\mathbf{T}_{2,\frac{1}{5}}\mathbf{M}$$

$$\mathbf{P}_{121} = \mathbf{P}_{111} = \mathbf{P}_{101} = \mathbf{Q}_{111} = \mathbf{Q}_{11\frac{1}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{P}_{112} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{P}_{212} = \mathbf{P}_{201} = \mathbf{Q}_{211} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{P}_{221} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{Q}_{2\frac{11}{55}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{T} = \ \mathbf{T}_{1,1} = \begin{bmatrix} \mathbf{T}(0) \\ \mathbf{T}\left(\frac{1}{2}\right) \\ \mathbf{T}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{T}_{2,\frac{1}{2}} = \begin{bmatrix} \mathbf{T}_{2,\frac{1}{2}}(0) \\ \mathbf{T}_{2,\frac{1}{2}}\left(\frac{1}{2}\right) \\ \mathbf{T}_{2,\frac{1}{2}}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{16} \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix},$$

$$\mathbf{T}_{2,\frac{1}{5}} = \begin{bmatrix} \mathbf{T}_{2,\frac{1}{5}}(0) \\ \mathbf{T}_{2,\frac{1}{5}}\left(\frac{1}{2}\right) \\ \mathbf{T}_{2,\frac{1}{5}}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{10} & \frac{1}{100} \\ 1 & \frac{1}{5} & \frac{1}{25} \end{bmatrix}$$

$$\hat{\mathbf{T}} = \hat{\mathbf{T}}_{1,1} = \hat{\mathbf{T}}_{2,1} = \begin{bmatrix} \mathbf{T}(0) & 0 & 0 \\ 0 & \mathbf{T}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \mathbf{T}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\hat{\mathbf{T}}_{2,\frac{1}{5}} = \begin{bmatrix} \mathbf{T}_{2,\frac{1}{5}}(0) & 0 & 0 \\ 0 & \mathbf{T}_{2,\frac{1}{5}}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \mathbf{T}_{2,\frac{1}{5}}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{10} & \frac{1}{100} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{25} \end{bmatrix}$$

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & 0 & \dots & 0 \\ 0 & \mathbf{M} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{M} \end{bmatrix}, \ \hat{\mathbf{C}}_r = \begin{bmatrix} \mathbf{C}_r & 0 & \dots & 0 \\ 0 & \mathbf{C}_r & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{C}_r \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \ \mathbf{C}_1 = [a \quad b \quad c]^T, \mathbf{C}_2 = [k \quad l \quad m]^T, \ \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}, \mathbf{G}_1 = \begin{bmatrix} 11 & \frac{141}{16} & 7 \end{bmatrix}^T, \mathbf{G}_2 = \begin{bmatrix} 3 & \frac{1281}{400} & \frac{44}{25} \end{bmatrix}^T$$

$$W_{11} = \begin{bmatrix} k+m+1 & 1 & k+m+3 \\ k+\frac{1}{2}l+\frac{5}{4}m+1 & \frac{1}{2}k+\frac{1}{4}l+\frac{5}{8}m+\frac{3}{2} & \frac{5}{4}k+\frac{5}{8}l+\frac{25}{16}m+\frac{17}{4} \\ k+l+2m+1 & k+l+2m+2 & 2k+2l+4m+6 \end{bmatrix}$$

$$W_{12} = \begin{bmatrix} a+c & 0 & a+c \\ a+\frac{1}{2}b+\frac{5}{4}c & \frac{1}{4}a+\frac{1}{8}b+\frac{5}{16}c-\frac{1}{2} & \frac{17}{16}a+\frac{17}{32}b+\frac{85}{64}c-\frac{1}{2} \\ a+b+2c & \frac{1}{2}a+\frac{1}{2}b+c-1 & \frac{5}{4}a+\frac{5}{4}b+\frac{5}{2}c-2 \end{bmatrix}$$

$$W_{21} = \begin{bmatrix} a+c+1 & 0 & a+c+1 \\ a+\frac{1}{2}b+\frac{5}{4}c+1 & \frac{1}{2}a+\frac{1}{4}b+\frac{5}{8}c+\frac{1}{2} & \frac{5}{4}a+\frac{5}{8}b+\frac{25}{16}c+\frac{9}{4} \\ a+b+2c+1 & a+b+2c+1 & 2a+2b+4c+4 \end{bmatrix}$$

$$W_{22} = \begin{bmatrix} -k-m & 1 & -k-m \\ -k-\frac{1}{10}l-\frac{101}{100}m & 1-\frac{1}{100}l-\frac{101}{1000}m-\frac{1}{10}k & 1-\frac{101}{1000}l-\frac{10\,201}{10\,000}m-\frac{101}{100}k \\ -k-\frac{1}{5}l-\frac{26}{25}m & 1-\frac{1}{25}l-\frac{26}{125}m-\frac{1}{5}k & 2-\frac{26}{125}l-\frac{676}{625}m-\frac{26}{25}k \end{bmatrix}$$

$$[\hat{\mathbf{W}}] = \begin{bmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{21} & \hat{W}_{22} \end{bmatrix} = \begin{bmatrix} k+m+1 & 1 & k+m+3 & a+c & 0 & a+c \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ a+c+1 & 0 & a+c+1 & -k-m & 1 & -k-m \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\hat{\mathbf{G}} = \begin{bmatrix} 11 & \mathbf{-3} & \mathbf{0} & 3 & \frac{1281}{400} & \mathbf{-2} \end{bmatrix}^T$$

where

$$\begin{aligned} z_1 &= a+\tfrac{1}{2}b+\tfrac{5}{4}c+1 \\ z_2 &= \tfrac{1}{2}a+\tfrac{1}{4}b+\tfrac{5}{8}c+\tfrac{1}{2} \\ z_3 &= \tfrac{5}{4}a+\tfrac{5}{8}b+\tfrac{25}{16}c+\tfrac{9}{4} \\ z_4 &= -k-\tfrac{1}{10}l-\tfrac{101}{100}m \\ z_5 &= 1-\tfrac{1}{100}l-\tfrac{101}{1000}m-\tfrac{1}{10}k \\ z_6 &= 1-\tfrac{101}{1000}l-\tfrac{10\,201}{10\,000}m-\tfrac{101}{100}k. \end{aligned}$$

From Eq. (3.16), the matrix form for initial condition is

$$[\mathbf{V}_{11};\delta_{11}] = [1 \quad 0 \quad 1 \quad ; \quad \mathbf{-3}],$$
$$[\mathbf{V}'_{11};\delta_{11}] = [0 \quad 1 \quad 0 \quad ; \quad \mathbf{0}],$$
$$[\mathbf{V}_{12};\delta_{12}] = [1 \quad 0 \quad 1 \quad ; \quad \mathbf{-2}]$$

Thus, the new augmented matrix $[\hat{\mathbf{W}};\hat{\mathbf{G}}]$ for the problem is gained. Solving this system, the Fibonacci coefficients matrix are found as

$$\mathbf{C} = [-4 \quad 0 \quad 1 - 2 \quad 1 \quad 0]^T$$

where

$$\mathbf{C}_1 = [-4 \quad 0 \quad 1]^T, \ \mathbf{C}_2 = [-2 \quad 1 \quad 0]^T.$$

The approximate solutions for $N = 2$ in terms of the Fibonacci polynomials are obtained as

$u_1(x) = x^2 - 3$and $u_2(x) = x - 2$.

Example 2 [35]. Assume that the following differential equation system

$$\begin{cases} u_1'(x) + u_1(x) + e^{-x}\cos\left(\frac{x}{2}\right)u_2\left(\frac{x}{2}\right) + 2e^{-(3/4)x}\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{4}\right)u_1\left(\frac{x}{4}\right) = 0 \\ u_2'(x) - e^x u_1^2\left(\frac{x}{2}\right) + u_2^2\left(\frac{x}{2}\right) = 0 \\ u_1(0) = 1, u_2(0) = 0 \end{cases} \tag{5.2}$$

Table 1 Numerical comparison of the error functions $E_{1,N}$ and $E_{2,N}$ at the different values of N for Example 2.

x	Adomian decomposition method [35], u_1			The proposed method, u_1		
	$E_{1,1}$	$E_{1,2}$	$E_{1,3}$	$E_{1,1}$	$E_{1,2}$	$E_{1,3}$
0.2	1.144×10^{-2}	4.432×10^{-4}	1.900×10^{-5}	2.410×10^{-3}	4.348×10^{-3}	9.376×10^{-4}
0.4	4.990×10^{-2}	4.274×10^{-3}	3.656×10^{-4}	1.740×10^{-2}	9.631×10^{-3}	1.014×10^{-3}
0.6	4.185×10^{-1}	1.643×10^{-2}	2.119×10^{-3}	5.295×10^{-2}	7.879×10^{-3}	1.976×10^{-4}
0.8	2.171×10^{-1}	4.274×10^{-2}	7.420×10^{-2}	1.130×10^{-1}	4.902×10^{-3}	8.460×10^{-4}
1	3.437×10^{-1}	8.925×10^{-2}	1.960×10^{-2}	1.987×10^{-1}	2.978×10^{-2}	1.061×10^{-2}
x	Adomian decomposition method [35], u_2			The proposed method, u_2		
	$E_{2,1}$	$E_{2,2}$	$E_{2,3}$	$E_{2,1}$	$E_{2,2}$	$E_{2,3}$
0.2	2.273×10^{-2}	5.174×10^{-4}	1.670×10^{-5}	1.330×10^{-3}	2.890×10^{-3}	1.548×10^{-5}
0.4	1.024×10^{-1}	5.840×10^{-3}	1.790×10^{-4}	1.058×10^{-2}	6.304×10^{-3}	3.048×10^{-4}
0.6	2.575×10^{-1}	2.630×10^{-2}	3.282×10^{-4}	3.535×10^{-2}	2.635×10^{-3}	9.485×10^{-4}
0.8	5.082×10^{-1}	8.022×10^{-2}	1.276×10^{-3}	8.264×10^{-2}	1.510×10^{-3}	1.408×10^{-3}
1	8.768×10^{-1}	1.965×10^{-1}	1.015×10^{-2}	1.585×10^{-1}	5.299×10^{-2}	2.475×10^{-4}

Table 2 Numerical results of the maximum error $E_{1,N}$ at the different values of N for Example 2.

N	2	5	8	11
$E_{1,N}$	2.978×10^{-2}	4.760×10^{-5}	2.894×10^{-8}	3.031×10^{-12}

Table 3 Numerical results of the maximum error $E_{2,N}$ at the different values of N for Example 2

N	2	5	8	11
$E_{2,N}$	5.299×10^{-2}	1.230×10^{-5}	3.037×10^{-9}	9.070×10^{-14}

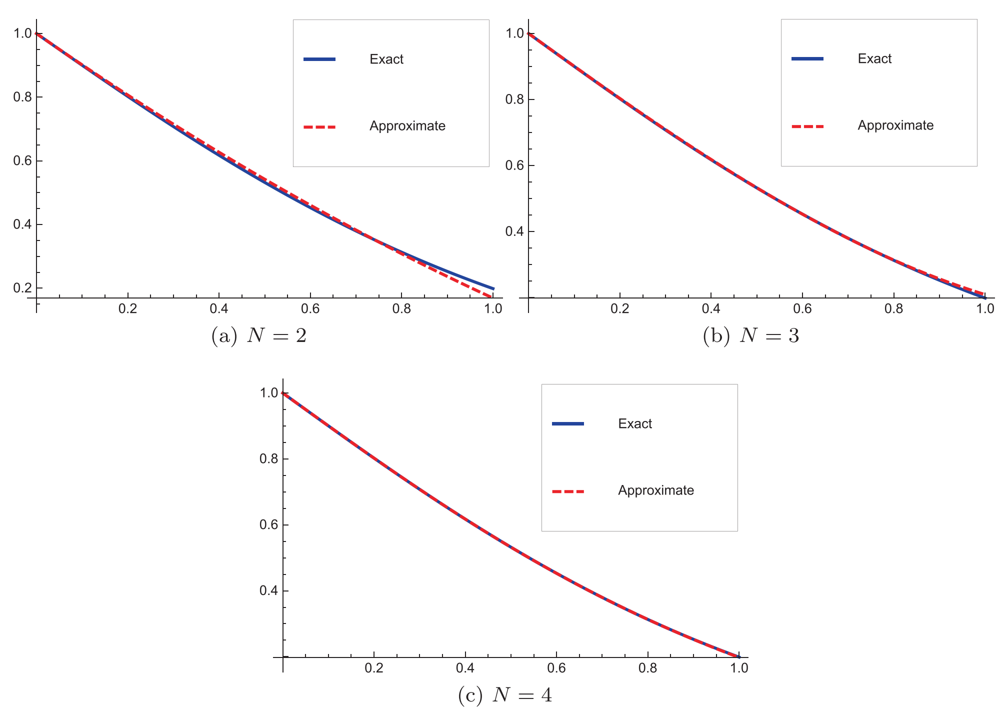


Fig. 1 Graphical comparison of the exact and approximate solutions for u_1 when $N = 2, 3, 4$ for Example 2.

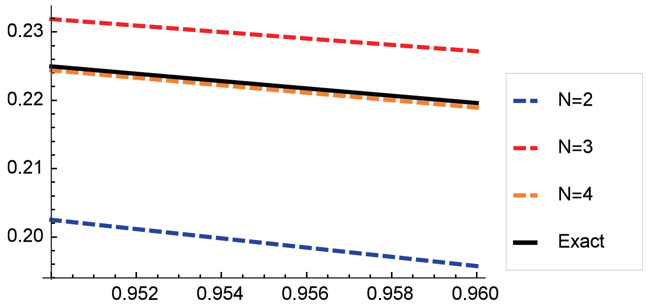


Fig. 2 The zoomed graphical comparison of the exact and approximate solutions for u_1 when $N = 2, 3, 4$ for Example 2.

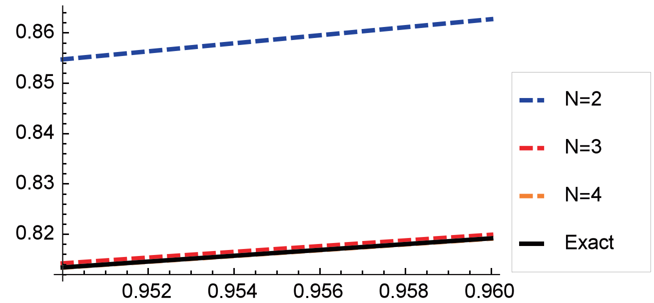


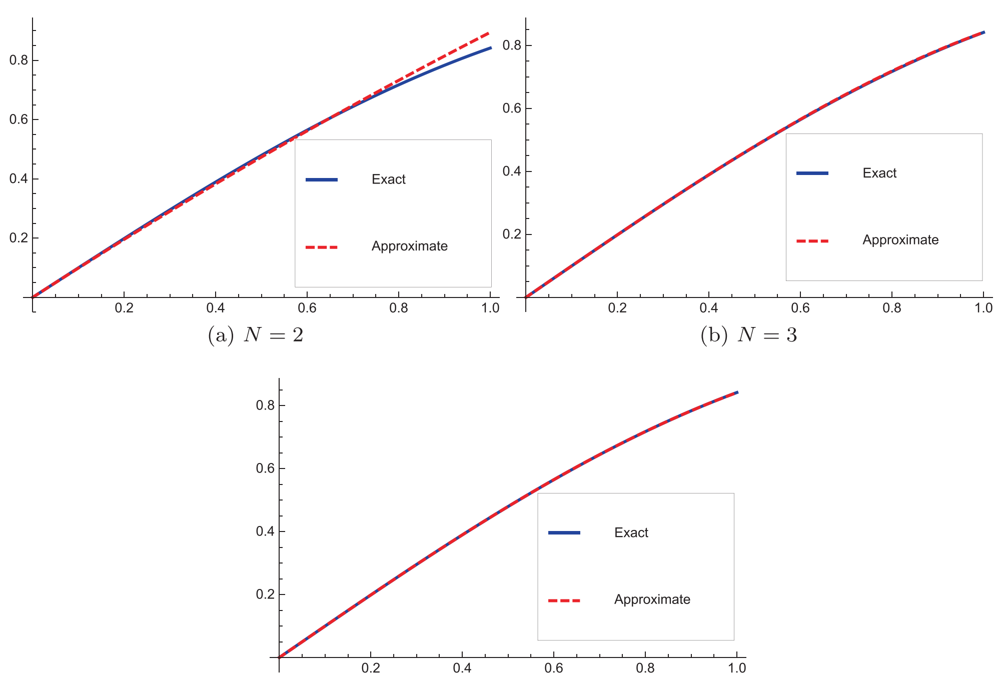
Fig. 4 The zoomed graphical comparison of the exact and approximate solutions for u_2 when $N = 2, 3, 4$ for Example 2.

The exact solution of Eq. (5.2) is given by $u_1(x) = e^{-x}\cos(x), u_2(x) = \sin(x)$. Table 1 presents the numerical values of error functions given in Eq. (4.1), Eq. (4.2) and a numerical comparison of proposed method with Adomian decomposition method [35] for Eq. (5.2) when $N = 1, 2$ and 3. Also, Table 2 and Table 3 show the numerical values of the maximum absolute error. In Fig. 1 and Fig. 3, it is presented that the graphical comparison of approximate and exact solutions obtained by the proposed method for u_1 and u_2 when $N = 2, 3$ and 4. Besides, in Fig. 2 and Fig. 4, it is given that graphics of the exact and approximate solutions obtained by the presented method in the interval (0.95, 0.96) when $N = 2, 3$ and 4.

Example 3. Consider that the following differential equation system

$$\begin{cases} -u_1''(x) + u_2''(x) + xu_1'(x) + u_1\left(\frac{x}{2}\right)u_2\left(\frac{x}{5}\right) & = & g_1(x) \\ xu_2''(x) + u_1''(x) + u_2^2(x) + u_2\left(\frac{x}{2}\right)u_1'\left(\frac{x}{10}\right) & = & g_2(x) \\ u_1(0) = 1, u_2(0) = 1 \\ u_1'(0) = 2, u_2'(0) = -2 \end{cases} \tag{5.3}$$

The exact solution of Eq. (5.3) is given by $u_1(x) = e^{2x}$, $u_2(x) = e^{-2x}$. Here, $g_1(x) = -2e^{2x}x + e^{\frac{3x}{5}} + 4e^{-2x}, g_2(x) = 4e^{-2x}(x + e^{4x}) + 2e^{-\frac{4x}{5}} + e^{-4x}$. Table 4 presents the numerical values of error function given in Eq. (4.1) and Eq. (4.2). Also, Table 5 and Table 6 show the numerical values of the maximum absolute error. In Fig. 5 and Fig. 7, it is shown that the graphical comparison of approximate and exact solutions obtained by the proposed method for u_1 and u_2 when $N = 3, 4$ and 5. Besides, in Fig. 6 and Fig. 8, it is given that graphics of the exact and approximate solutions obtained by the presented method in the interval (0.95, 0.96) when $N = 3, 4$ and 5.



(a) $N = 2$ (b) $N = 3$

(c) $N = 4$

Fig. 3 Graphical comparison of the exact and approximate solutions for u_2 when $N = 2, 3, 4$ for Example 2.

Table 4 Numerical comparison of the error functions $E_{1,N}$ and $E_{2,N}$ at the different values of N for Example 3.

x	u_1			u_2		
	$E_{1,3}$	$E_{1,5}$	$E_{1,8}$	$E_{2,3}$	$E_{2,5}$	$E_{2,8}$
0.2	3.335×10^{-3}	1.010×10^{-4}	9.805×10^{-8}	1.792×10^{-3}	3.414×10^{-5}	2.750×10^{-8}
0.4	1.573×10^{-2}	2.257×10^{-4}	2.102×10^{-7}	7.574×10^{-3}	7.175×10^{-5}	6.443×10^{-8}
0.6	9.195×10^{-3}	3.851×10^{-4}	3.252×10^{-7}	5.855×10^{-3}	1.044×10^{-4}	1.124×10^{-7}
0.8	1.028×10^{-1}	1.680×10^{-4}	4.148×10^{-7}	2.666×10^{-2}	5.523×10^{-5}	1.688×10^{-7}
1	4.940×10^{-1}	1.266×10^{-2}	1.547×10^{-5}	1.212×10^{-1}	2.641×10^{-3}	2.680×10^{-6}

Table 5 Numerical results of the maximum error $E_{1,N}$ at the different values of N for Example 3

N	2	5	8	11
$E_{1,N}$	2.389×10^{0}	1.266×10^{-2}	1.547×10^{-5}	5.651×10^{-9}

Table 6 Numerical results of the maximum error $E_{2,N}$ at the different values of N for Example 3

N	2	5	8	11
$E_{2,N}$	8.646×10^{-1}	2.641×10^{-3}	2.680×10^{-6}	9.837×10^{-10}

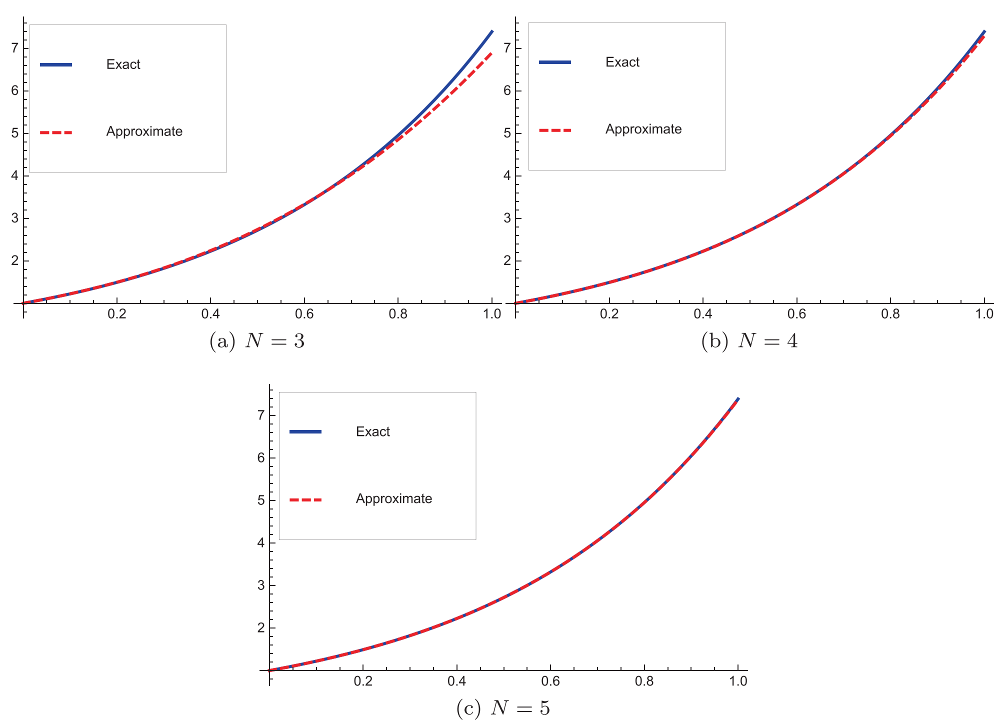


Fig. 5 Graphical comparison of the exact and approximate solutions for u_1 when $N = 3, 4, 5$ for Example 3.

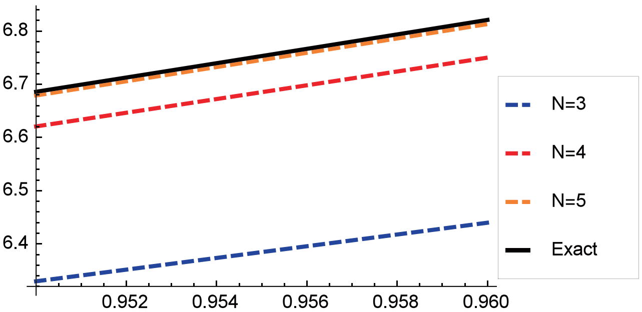


Fig. 6 The zoomed graphical comparison of the exact and approximate solutions for u_1 when $N = 3, 4, 5$ for Example 3.

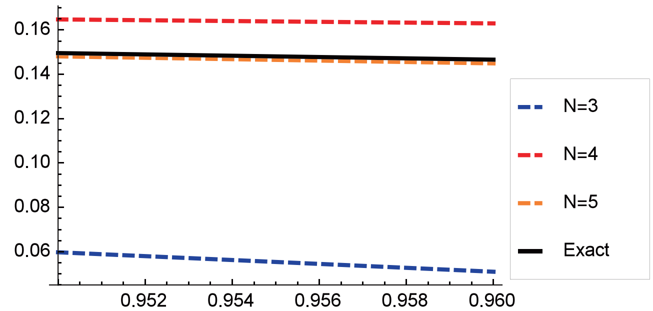


Fig. 8 The zoomed graphical comparison of the exact and approximate solutions for u_2 when $N = 3, 4, 5$ for Example 3.

6. Conclusions

In this study, the Fibonacci collocation method was used for solving a class of systems of nonlinear Pantograph differential equations. The efficiency and accuracy of the method with three different examples are shown. The obtained approximate and error results are compared with ones obtained with Adomian decomposition method. As a result of these comparisons, it can be said that the method is very effective to obtain approximate solution systems of nonlinear Pantograph differential equations. The given tables and graphics show that when it is increased that the number of N, the approximate solutions converge the exact ones. Besides, as seen in Example 1, for problems whose analytical solution is polynomial, it is possible to obtain the exact solution using the presented method. The other advantage of the method is that all the computations can be calculated in a short time with computer software. In the future, it is planned to apply the method to systems of fractional differential equations.

Author Contribution

Authors completed this study and wrote the manuscript. Also, authors read and approved the final manuscript.

Funding Information

There are no funders to report for this submission.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

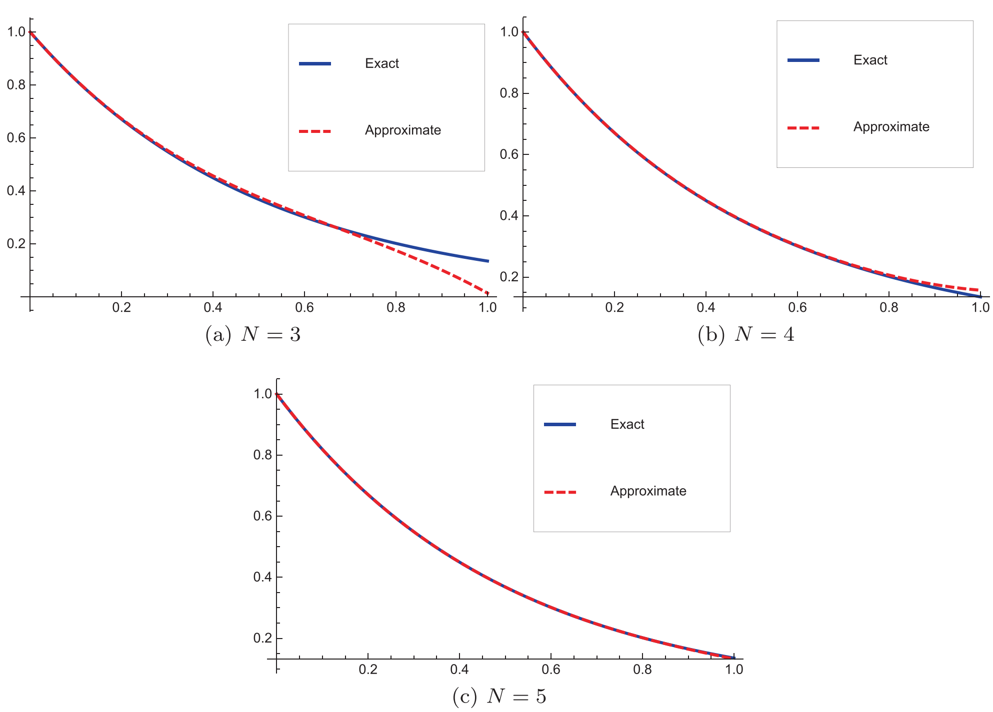


Fig. 7 Graphical comparison of the exact and approximate solutions for u_2 when $N = 3, 4, 5$ for Example 3.

References

[1] C. Ravichandran, K. Logeswari, F. Jarad, New results on existence in the framework of Atangana-Baleanu derivative for fractional integro-differential equations, Chaos Solitons Fract. 125 (2019) 194–200.

[2] A. Kumar, H.V.S. Chauhan, C. Ravichandran, K.S. Nisar, D. Baleanu, Existence of solutions of non-autonomous fractional differential equations with integral impulse condition, Adv. Differ. Eqs. 2020 (1) (2020) 1–14.

[3] K. Jothimani, N. Valliammal, C. Ravichandran, Existence result for a neutral fractional integro-differential equation with state dependent delay, J. Appl. Nonlinear Dyn. 7 (4) (2018) 371–381.

[4] N. Valliammal, C. Ravichandran, Results on fractional neutral integro-differential systems with state-dependent delay in Banach spaces, Nonlinear Stud. 25 (1) (2018).

[5] S.K. Panda, C. Ravichandran, B. Hazarika, Results on system of Atangana-Baleanu fractional order Willis aneurysm and nonlinear singularly perturbed boundary value problems, Chaos Solitons Fract. 142 (2021) 110390.

[6] K.S. Nisar, K. Jothimani, K. Kaliraj, C. Ravichandran, An analysis of controllability results for nonlinear Hilfer neutral fractional derivatives with non-dense domain, Chaos Solitons Fract. 146 (2021) 110915.

[7] C. Ravichandran, N. Valliammal, J.J. Nieto, New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach spaces, J. Franklin Inst. 356 (3) (2019) 1535–1565.

[8] S. Kumar, A. Kumar, A.M. Wazwaz, New exact solitary wave solutions of the strain wave equation in microstructured solids via the generalized exponential rational function method, Eur. Phys. J. Plus 135 (11) (2020) 1–17.

[9] S. Kumar, M. Niwas, A.M. Wazwaz, Lie symmetry analysis, exact analytical solutions and dynamics of solitons for $(2+1)$-dimensional NNV equations, Phys. Scr. 95 (9) (2020) 095204.

[10] S. Kumar, A. Kumar, Lie symmetry reductions and group invariant solutions of $(2+1)$-dimensional modified Veronese web equation, Nonlinear Dyn. 98 (3) (2019) 1891–1903.

[11] S. Kumar, H. Almusawa, A. Kumar, Some more closed-form invariant solutions and dynamical behavior of multiple solitons for the $(2+1)$-dimensional rdDym equation using the Lie symmetry approach, Res. Phys. 24 (2021) 104201.

[12] B. Ghanbari, S. Kumar, M. Niwas, D. Baleanu, The Lie symmetry analysis and exact Jacobi elliptic solutions for the Kawahara-KdV type equations, Res. Phys. 23 (2021) 104006.

[13] B. Ghanbari, K.S. Nisar, Determining new soliton solutions for a generalized nonlinear evolution equation using an effective analytical method, Alexandr. Eng. J. 59 (5) (2020) 3171–3179.

[14] K. Munusamy, C. Ravichandran, K.S. Nisar, B. Ghanbari, Existence of solutions for some functional integrodifferential equations with nonlocal conditions, Math. Methods Appl. Sci. 43 (17) (2020) 10319–10331.

[15] B. Ghanbari, K.S. Nisar, M. Aldhaifallah, Abundant solitary wave solutions to an extended nonlinear Schrödinger's equation with conformable derivative using an efficient integration method, Adv. Differ. Eqs. 2020 (1) (2020) 1–25.

[16] V. Vijayakumar, R. Udhayakumar, A new exploration on existence of Sobolev-type Hilfer fractional neutral integro-differential equations with infinite delay, Numer. Methods Partial Differ. Eqs. 37 (1) (2021) 750–766.

[17] W.K. Williams, V. Vijayakumar, R. Udhayakumar, K.S. Nisar, A new study on existence and uniqueness of nonlocal fractional delay differential systems of order $1 < r < 2$ in Banach spaces, Numer. Methods Partial Differ. Eqs. 37 (2) (2020) 949–961.

[18] C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, K.S. Nisar, A discussion on the approximate controllability of Hilfer fractional neutral stochastic integro-differential systems, Chaos Solitons Fract. 142 (2021) 110472.

[19] M. Mohan Raja, V. Vijayakumar, R. Udhayakumar, K.S. Nisar, Results on existence and controllability results for fractional evolution inclusions of order $1 < r < 2$ with Clarke's subdifferential type, Numer. Methods Partial Differ. Eqs. early view, in press

[20] M.M. Raja, V. Vijayakumar, R. Udhayakumar, Results on the existence and controllability of fractional integro-differential system of order $1 < r < 2$ via measure of noncompactness, Chaos Solitons Fract. 139 (2020) 110299.

[21] M.M. Raja, V. Vijayakumar, R. Udhayakumar, Y. Zhou, A new approach on the approximate controllability of fractional differential evolution equations of order $1 < r < 2$ in Hilbert spaces, Chaos Solitons Fract. 141 (2020) 110310.

[22] J.H. He, H. Latifizadeh, A general numerical algorithm for nonlinear differential equations by the variational iteration method, Int. J. Numer. Methods Heat Fluid Flow 30 (11) (2020) 4797–4810.

[23] D. Rani, V. Mishra, Numerical inverse Laplace transform based on Bernoulli polynomials operational matrix for solving nonlinear differential equations, Res. Phys. 16 (2020) 102836.

[24] Z. Odibat, An optimized decomposition method for nonlinear ordinary and partial differential equations, Physica A 541 (2020) 123323.

[25] Z. Odibat, An improved optimal homotopy analysis algorithm for nonlinear differential equations, J. Math. Anal. Appl. 488 (2) (2020) 124089.

[26] A. Kurt, S. Yalçinbas, M. Sezer, Fibonacci collocation method for solving linear differential-difference equations, Math. Comput. Appl. 18 (3) (2013) 448–458.

[27] A. Kurt, S. Yalçinbas, M. Sezer, Fibonacci collocation method for solving high-order linear Fredholm integro-differential-difference equations, Int. J. Math. Math. Sci. 2013 (2013).

[28] F. Mirzaee, S.F. Hoseini, Solving systems of linear Fredholm integro-differential equations with Fibonacci polynomials, Ain Shams Eng. J. 5 (1) (2014) 271–283.

[29] F. Mirzaee, S.F. Hoseini, Solving singularly perturbed differential-difference equations arising in science and engineering with Fibonacci polynomials, Res. Phys. 3 (2013) 134–141.

[30] F. Mirzaee, S.F. Hoseini, A new collocation approach for solving systems of high-order linear Volterra integro-differential equations with variable coefficients, Appl. Math. Comput. 311 (2017) 272–282.

[31] H. Smith, An introduction to delay differential equations with applications to the life sciences, vol. 57, Springer Science & Business Media, New York, 2010.

[32] T. Erneux, Applied delay differential equations, vol. 3, Springer Science & Business Media, New York, 2009.

[33] S. Falcon, A. Plaza, The k-Fibonacci sequence and the Pascal 2-triangle, Chaos Solitons Fract. 33 (1) (2007) 38–49.

[34] S. Falcon, A. Plaza, On k-Fibonacci sequences and polynomials and their derivatives, Chaos Solitons Fract. 39 (3) (2009) 1005–1019.

[35] S. Widatalla, M.A. Koroma, Approximation algorithm for a system of pantograph equations, J. Appl. Math. 2012 (2012).