A SOLUTION METHOD FOR INTEGRO-DIFFERENTIAL EQUATIONS OF CONFORMABLE FRACTIONAL DERIVATIVE

by

Mustafa BAYRAM[a*], Veysel Fuat HATIPOGLU[b], Sertan ALKAN[c], and Sebahat Ebru DAS[d]

[a] Department of Computer Engineering, Gelisim University, Istanbul, Turkey
[b] Department of Business Administration, Mugla Sitki Kocman University, Mugla, Turkey
[c] Department of Computer Engineering, Iskenderun Technical University, Hatay, Turkey
[d] Department of Mathematics, Yildiz Technical University, Istanbul, Turkey

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The aim of this work is to determine an approximate solution of a fractional order Volterra-Fredholm integro-differential equation using by the Sinc-collocation method. Conformable derivative is considered for the fractional derivatives. Some numerical examples having exact solutions are approximately solved. The comparisons of the exact and the approximate solutions of the examples are presented both in tables and graphical forms.

Key words: *Sinc-collocation method, conformable fractional derivative, Volterra-Fredholm integro-differential equation*

Introduction

In recent years, numerous problems from physics, mathematics, biology, chemistry, engineering, and other various sciences involving fractional calculus has been studied by many authors. Several numerical methods for solving linear and non-linear fractional integro-differential equations based on Riemann-Liouville and Caputo derivative have been presented. Many numerical methods such as, wavelets [1, 2], Adomian decomposition method [3, 4], homotopy perturbation method [5], homotopy analysis method [6], and variational iteration method [7] have been used to solve fractional integral equations and integro-differential equations.

In this work, we consider the following form of the fractional Volterra-Fredholm integro-differential equation:

$$\mu_\alpha(x)y^{(\alpha)}(x)=f(x)+\lambda_1\int_\alpha^x K_1(x,t)y(t)\mathrm{d}t+\lambda_2\int_\alpha^b K_2(x,t)y(t)\mathrm{d}t \tag{1}$$

with the boundary conditions:

$$y(a)=0,\ \ y(b)=0 \tag{2}$$

where $y(\alpha)$ is the conformable fractional derivative for $1<\alpha\le 2$. Here $T_\alpha(f)$ where $t>0$, $\alpha\in(0,1)$ be understood as conformable fractional derivative which was defined in [8]. Some properties of the conformable fractional derivative are given in the next section.

* Corresponding author, e-mail: mbayram@gelisim.edu.tr

Preliminaries

In this section, the fundamental theorems and definitions are introduced. Readers can look for more details in [9-16].

Definition 1. Let $\alpha \in (n, n+1]$ and *f* be an *n*-differentiable function at *t*, where $\mathrm{t} > 0$, Then the conformable fractional derivative of *f* of order *α* is defined:

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{(\lceil \alpha \rceil - 1)}\left[t + \varepsilon t^{(\lceil \alpha \rceil - \alpha)}\right] - f^{(\lceil \alpha \rceil - 1)}(t)}{\varepsilon} \tag{3}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to *α*.

Remark 1. As a consequence of *Definition 1*, one can easily show that:

$$T_\alpha(f)(t) = t^{(\lceil \alpha \rceil - \alpha)} f^{\lceil \alpha \rceil}(t) \tag{4}$$

where $\alpha \in (n, n+1]$ and *f* is $(n+1)$ differentiable at $t > 0$.

Theorem 1. Let $\alpha \in (n, n+1]$ and *f, g* be *α*-differentiable at a point $t > 0$. Then:
(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$,
(2) $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$,
(3) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$,
(4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$, and
(5) $T_\alpha(f/g) = [gT_\alpha(f) + fT_\alpha(g)]/g^2$

Definition 2. The function:

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{5}$$

is called the sinc (sinc cardinal) function.

Definition 3. The translated sinc function with space points are defined:

$$S(k,h)(x) = \mathrm{sinc}\left(\frac{x - kh}{h}\right) = \begin{cases} \dfrac{\sin\left(\pi \dfrac{x - kh}{h}\right)}{\pi \dfrac{x - kh}{h}}, & x \neq kh \\ 1, & x = kh \end{cases} \tag{6}$$

where $h > 0$ and $k = 0, \pm 1, \pm 2, \ldots$

For establishing the approximation on (a, b), the conformal map is defined:

$$\phi(z) = In\left(\frac{z - a}{b - z}\right) \tag{7}$$

Here, the basis functions are attained using the composite translated sinc functions given:

$$S_k(z) = S(k,h)(z) o \phi(z) = \mathrm{sinc}\left[\frac{\phi(z) - kh}{h}\right] \tag{8}$$

$$z = \phi^{-1}(w) = \frac{a + be^w}{1 + e^w} \tag{9}$$

is the inverse map of $w=\phi(z)$. The sinc grid points $z_k \in (a,b)$ in D_E are real numbers, so that they can denoted by x_k. The notation o denotes the Hadamard matrix multiplication. For the evenly spaced points $\{kh\}_{k=-\infty}^{\infty}$, the image corresponding to these points is defined:

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, \quad k=0,\pm1,\pm2,\ldots \tag{10}$$

The sinc-collocation method

Consider the approximate solution of eq. (1) is given:

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \quad n = M+N+1 \tag{11}$$

Here, $S_k(x)$ is the composite function of $S(k,h)$ and $\phi(x)$ for some fixed step size h. The unknown coefficients c_k in eq. (11) are obtained with the help of the sinc-collocation method (SCM).

Theorem 2. The conformable fractional derivative of $y_n(x)$ is:

$$y_n^{(\alpha)}(x) = \sum_{k=-M}^{N} c_k x^{2-\alpha}\left\{\phi''(x)\frac{\mathrm{d}}{\mathrm{d}\phi}S_k(x) + [\phi'(x)]^2\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}S_k(x)\right\}, \quad 1<\alpha\le 2 \tag{12}$$

Proof. The conformable fractional derivative of $y_n(x)$ can also be written from eq. (11):

$$y_n^{(\alpha)}(x) = \sum_{k=-M}^{N} c_k S_k^{(\alpha)}(x) \tag{13}$$

with the help of eqs. (3) and (4), we have:

$$S_k^{(\alpha)}(x) = x^{2-\alpha} S_k''(x) \tag{14}$$

When we write eq. (14) in eq. (13), we find that:

$$y_n^{(\alpha)}(x) = \sum_{k=-M}^{N} c_k x^{2-\alpha}\left\{\phi''(x)\frac{\mathrm{d}}{\mathrm{d}\phi}S_k(x) + [\phi'(x)]^2\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}S_k(x)\right\}$$

With the aid of Theorem 2.13 in [17], the below two lemmas are presented.

Lemma 1. The following relation provides:

$$\int_a^{x_j} K_1(x,t)y(t)\,\mathrm{d}t \approx h\sum_{k=-M}^{N}\delta_{jk}^{(-1)}\frac{K_1(x_j,t_k)}{\phi'(t_k)}y_k \tag{15}$$

where

$$\sigma_{jk} = \int_0^{j-k}\frac{\sin \pi t}{\pi t}\mathrm{d}t, \quad \delta_{jk}^{(-1)} = \frac{1}{2}+\sigma_{jk} \tag{16}$$

and y_k is the approximate value of $y(t_k)$.

Lemma 2. The following relation provides

$$\int_a^{b} K_2(x,t)y(t)\,\mathrm{d}t \approx h\sum_{k=-M}^{N}\frac{K_2(x,t_k)}{\phi'(t_k)}y_k \tag{17}$$

where y_k is the approximate value of $y(t_k)$.

Replacing each term of eq. (1) with the approach defined in eqs. (11)-(17) and the producing with $\{(1/\phi')^2\}$, we determine:

$$\sum_{k=-M}^{N}\left\{c_k\left[\sum_{i=1}^{2}g_i(x)\frac{\mathrm{d}^i}{\mathrm{d}\phi^i}S_k+g_3(x)\delta_{jk}^{(-1)}\frac{K_1(x,t_k)}{\phi'(t_k)}+g_4(x)\frac{K_2(x,t_k)}{\phi'(t_k)}\right]\right\}=\left\{f(x)\left[\frac{1}{\phi'(x)}\right]^2\right\} \quad (18)$$

where

$$g_1(x)=-\left[\frac{1}{\phi'(x)}\right]', \quad g_2(x)=x^{2-\alpha}, \quad g_3(x)=-\lambda_1 h\left[\frac{1}{\phi'(x)}\right]^2, \quad g_4(x)=-\lambda_2 h\left[\frac{1}{\phi'(x)}\right]^2 \quad (19)$$

We know from [18] that:

$$\delta_{jk}^{(0)}=\delta_{kj}^{(0)}, \quad \delta_{jk}^{(1)}=-\delta_{kj}^{(1)}, \quad \delta_{jk}^{(2)}=\delta_{kj}^{(2)} \quad (20)$$

Theorem 3. Let us consider the boundary value problem eqs. (1) and (2). Then the discrete sinc-collocation system for determining the unknown coefficients $\{c_k\}_{k=-M}^{N}$ of the approximate solution is given:

$$\sum_{k=-M}^{N}\left\{c_k\left[\begin{array}{c}\frac{g_1(x_j)}{h^2}\delta_{jk}^{(2)}+\frac{g_2(x_j)}{h}\delta_{jk}^{(1)}+g_3(x_j)\delta_{jk}^{(-1)}\frac{K_1(x_j,t_k)}{\phi'(t_k)}+\\ +g_4(x_j)\frac{K_2(x_j,t_k)}{\phi'(t_k)}\end{array}\right]\right\}=\left\{f(x_j)\left[\frac{1}{\phi'(x_j)}\right]^2\right\} \quad (21)$$

for $j=-M,\ldots,N$.

Some notations are defined to rewrite eq. (21) in the matrix form. Let $D(y)$ be a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}),\ldots,y(x_N)$ and non-diagonal elements are zero:

$$E_1=\frac{K_1(x_j,t_k)}{\left[\phi'(x_j)\right]^2\phi'(t_k)}, \quad E_2=\frac{K_2(x_j,t_k)}{\left[\phi'(x_j)\right]^2\phi'(t_k)} \quad (22)$$

denote a matrix and also let $I^{(i)}$ denote the matrices:

$$I^{(i)}=\left[\delta_{jk}^{(i)}\right], \quad i=-1,0,1,2 \quad (23)$$

where D, G, E_1, E_2, $I^{(-1)}$, $I^{(0)}$, $I^{(1)}$, and $I^{(2)}$ are $n\times n$ matrices. By using the previous notations in eq. (21), we can represent it:

$$Ac=B \quad (24)$$

where

$$A=\sum_{i=1}^{2}\frac{1}{h^i}D(g_i)I^{(i)}-D(g_3)\left[E_1\,o\,I^{(-1)}\right]-D(g_4)E_2 \quad (25)$$

$$B=D\left[\frac{f}{(\phi')^2}\right]I \quad (26)$$

$$c=(c_{-M},c_{-M+1},\ldots,c_N)^T \quad (27)$$

The notation o in A denotes the Hadamard matrix multiplication. Finally, we can reach the approximate solution of eq. (21) after finding the unknown coefficients c_k in the system.

Numerical examples

In this section, SCM is applied to two different problems using *Mathematica10*. In each example, we consider $h = \pi/N^{1/2}$ and $N = M$.

Example 1. Let us take the following boundary value problem:

$$y^{(\alpha)}(x) = f(x) + \int_a^x K_1(x,t)y(t)\,\mathrm{d}t + \int_a^b K_2(x,t)y(t)\,\mathrm{d}t\,, \quad y(0)=0, \quad y(1)=0 \tag{28}$$

where

$$\alpha = \frac{7}{4}, \quad K_1(x,t) = \sin(x-t), \quad K_2(x,t) = \cos(x-t)$$

and

$$f(x) = 24 + 6x + 6x^{\frac{5}{4}} - 12x^2 - 12x^{\frac{9}{4}} - x^3 + x^4 -$$
$$-17\cos(1-x) - 30\cos(x) + 18\sin(1-x) + 18\sin(x)$$

This problem has an exact solution in the form of $(x) = x^3(1-x)$. The comparisons of the exact and the approximate solutions of the example are shown graphically for different N values in fig. 1. In addition, the approximate solution obtained with the aid of SCM of this problem is shown in tab. 1.

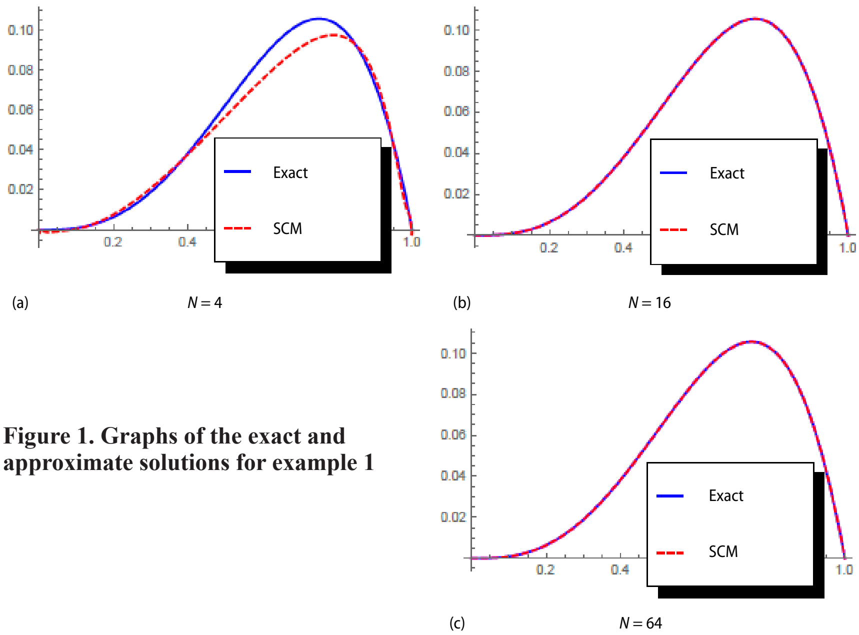


Figure 1. Graphs of the exact and approximate solutions for example 1

Example 2. Let us consider the fractional integro-differential equation:

$$y^{(\alpha)}(x) = f(x) - 2\int_a^x K_1(x,t)y(t)\,\mathrm{d}t - \int_a^b K_2(x,t)y(t)\,\mathrm{d}t, \quad y(0)=0, \quad y(1)=0 \tag{29}$$

Table 1. Errors between the exact and the approximate solution of *Example 1*

x	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
0.1	$4.763\cdot10^{-4}$	$2.250\cdot10^{-4}$	$4.387\cdot10^{-6}$	$8.397\cdot10^{-8}$	$3.403\cdot10^{-10}$
0.2	$1.212\cdot10^{-3}$	$5.622\cdot10^{-5}$	$1.356\cdot10^{-5}$	$8.274\cdot10^{-8}$	$1.479\cdot10^{-10}$
0.3	$1.903\cdot10^{-3}$	$3.566\cdot10^{-4}$	$2.956\cdot10^{-6}$	$1.653\cdot10^{-7}$	$1.253\cdot10^{-10}$
0.4	$5.898\cdot10^{-4}$	$2.867\cdot10^{-4}$	$2.459\cdot10^{-5}$	$2.490\cdot10^{-7}$	$7.088\cdot10^{-11}$
0.5	$5.797\cdot10^{-3}$	$5.590\cdot10^{-4}$	$1.965\cdot10^{-5}$	$1.690\cdot10^{-7}$	$1.793\cdot10^{-10}$
0.6	$1.091\cdot10^{-2}$	$1.360\cdot10^{-3}$	$5.734\cdot10^{-5}$	$3.868\cdot10^{-7}$	$1.744\cdot10^{-10}$
0.7	$1.179\cdot10^{-2}$	$8.753\cdot10^{-4}$	$1.851\cdot10^{-5}$	$6.084\cdot10^{-7}$	$5.926\cdot10^{-10}$
0.8	$5.200\cdot10^{-3}$	$7.183\cdot10^{-4}$	$2.811\cdot10^{-5}$	$7.760\cdot10^{-7}$	$9.909\cdot10^{-10}$
0.9	$3.554\cdot10^{-3}$	$2.366\cdot10^{-4}$	$3.453\cdot10^{-5}$	$3.566\cdot10^{-8}$	$9.636\cdot10^{-10}$

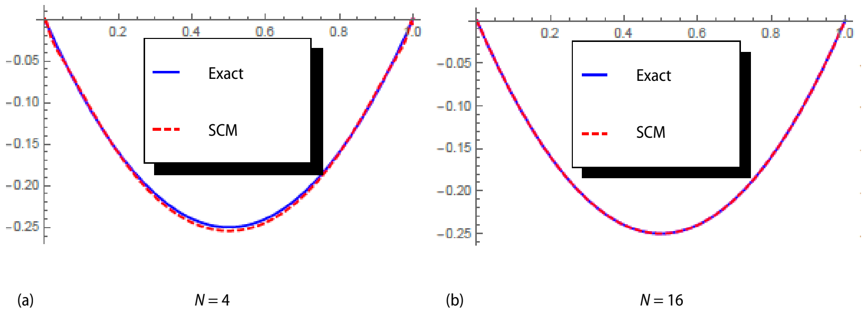
where

$$\alpha=\frac{3}{2},\quad K_1(x,t)=x-t,\quad K_2(x,t)=x-t^2$$

and

$$f(x)=\frac{1}{20}+2\sqrt{x}-\frac{x}{6}-\frac{x^3}{3}+\frac{x^4}{6}$$

The exact solution of this problem is $(x)=x(x-1)$. The numerical solutions determined by SCM of the problem are presented in tab. 2. Furthermore, the comparisons of the exact and approximate solutions are given graphically in fig. 2.



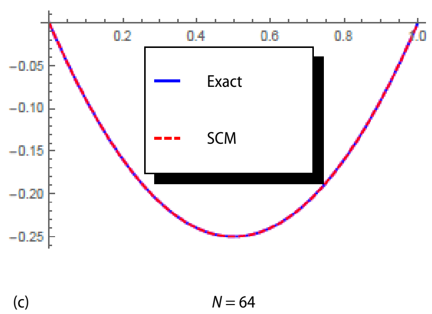


Figure 2. Graphs of the exact and approximate solutions for example 2

Table 2. Errors between the exact and the approximate solution of *Example 2*

x	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0.1	$2.188\cdot10^{-3}$	$1.585\cdot10^{-4}$	$4.404\cdot10^{-6}$	$4.734\cdot10^{-10}$	$6.423\cdot10^{-11}$
0.2	$1.016\cdot10^{-3}$	$2.076\cdot10^{-4}$	$2.901\cdot10^{-6}$	$5.654\cdot10^{-8}$	$5.517\cdot10^{-11}$
0.3	$3.717\cdot10^{-3}$	$9.061\cdot10^{-5}$	$4.063\cdot10^{-6}$	$4.285\cdot10^{-8}$	$1.952\cdot10^{-11}$
0.4	$4.403\cdot10^{-3}$	$2.216\cdot10^{-4}$	$3.256\cdot10^{-6}$	$1.410\cdot10^{-9}$	$1.696\cdot10^{-11}$
0.5	$4.432\cdot10^{-3}$	$2.276\cdot10^{-4}$	$3.880\cdot10^{-6}$	$1.355\cdot10^{-8}$	$4.421\cdot10^{-12}$
0.6	$4.404\cdot10^{-3}$	$2.185\cdot10^{-4}$	$3.116\cdot10^{-6}$	$4.850\cdot10^{-10}$	$1.719\cdot10^{-11}$
0.7	$3.721\cdot10^{-3}$	$8.618\cdot10^{-5}$	$4.171\cdot10^{-6}$	$4.260\cdot10^{-8}$	$1.986\cdot10^{-11}$
0.8	$1.026\cdot10^{-3}$	$2.101\cdot10^{-4}$	$2.845\cdot10^{-6}$	$5.649\cdot10^{-8}$	$5.502\cdot10^{-11}$
0.9	$2.168\cdot10^{-3}$	$1.605\cdot10^{-4}$	$4.412\cdot10^{-6}$	$5.358\cdot10^{-10}$	$6.401\cdot10^{-11}$

Conclusion

This work focused on the determination of an approximate solution of Volterra-Fredholm integro-differential equations introduced in eq. (1). The conformable derivative is taken as the fractional derivative. The SCM is applied to eq. (1). It can be easily seen that SCM gives good results for the eq. (1). Perspective of the conformable derivative sense, It can be seen that numerical solutions regarding to the proposed method are also approximated well like previous studies based on other fractional derivative definitions. As a result, we can say that SC algorithm is a powerful tool for obtaining the approximate solution of eq. (1).

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